

# Modeling the Physical World with Common Sense and Mathematics

David Hestenes

***Abstract:*** *Physics and mathematics are grounded in man's evolved ability to freely create mental models and use them to manage interactions with the natural world.*

The *Copernican Revolution* in science culminated in Newton's *Principia* (1687), which integrated astronomy and terrestrial physics into a single science of motion. Immanuel Kant (1787) saw this as a striking union of mathematical theory with empirical fact that bridged the traditional divide between rationalism and empiricism. So he proposed a comparable "*Copernican Revolution*" in philosophy to account for it [1]. Just as Copernicus shifted the center of the universe from earth to sun, Kant shifted the focus of epistemology from structure of the external world to structure of mind. His revolutionary insight was that our perceptions and thoughts are shaped by inherent structure of our minds. He argued that the fundamental laws of nature, like the truths of mathematics, are knowable precisely because they do not describe the world as it really is but rather prescribe the structure of the world as we experience it.

Though the scientific revolution has expanded in spectacular fashion to integrate physics and astronomy with chemistry and biology, Kant's revolution in philosophy has hardly progressed. My purpose here is to open a new stage in Kant's revolution by explaining how findings of cognitive science can be marshaled to create a new "*science of mind*" with testable predictions and explanations as required of any "true" science. I begin with a restatement of Kant's primary question: *What does the structure of science and mathematics tell us about how the human mind works?* In searching for answers my working hypothesis will be: *The primary cognitive activities in science and mathematics involve making, validating and applying conceptual models!* In a word, science and mathematics are about MODELING — making and using models!

This essay argues for a "MODELING THEORY of MIND" to guide the multifarious branches of cognitive science in research on the nature of mind and brain, and the design of conceptual tools for science and mathematics. Core principles are explained and supporting evidence is sketched, but the brush is necessarily broad. More details are given in [2,3,4], especially for application to physics teaching and learning.

## I. NEWTON'S MODELING GAME

Newton did much more than provide the first mathematical formulation of a scientific theory in his *Principia*; he also demonstrated how to relate it to empirical fact. Though Kant recognized revolutionary implications for epistemology in this impressive feat, physicists have overlooked it. The issue has been thoroughly explicated in [5] by framing Newtonian theory in terms of models and modeling. A central point is that theoretical principles like Newton's Laws cannot be tested or applied except by incorporating them in models.

Kant's insight can be explicated by noting that Newton linked up two distinct kinds of models: theoretical and empirical. A *theoretical model* derived from Newton's Laws predicts motions, while an *empirical model* derived from data describes a motion. A match between them explains a motion. In this way Newton explained Galileo's law of falling bodies and Kepler's three laws of planetary motion. Note the distinction between a theoretical Law (with a capital L) and an empirical law (with a lowercase l).

Comparison between theoretical and empirical models is such a standard practice of physicists since Newton that they seldom consider its profound epistemological implications. At its simplest, it involves creating an empirical model from data with a procedure often called “curve fitting.” That’s how Kepler’s laws were derived. It is an important technique in the search for empirical regularities that are both quantifiable and reproducible. In high energy physics data analysis has become so complex that a new research specialty has emerged to handle it. That research, often called “phenomenology,” is thus intermediate between theory and experiment.

For future analysis, it is worth noting that scientific work in all three domains is governed by definite but different rules; from mathematical rules for theorists, to measurement standards for experimentalists, to probability theory for phenomenologists. As Kant recognized, scientific objectivity requires strict adherence to rules. The question is: Where do the rules come from?

## II. FROM COMMON SENSE TO SCIENTIFIC THINKING

As we grow and learn through everyday experience, each of us develops a system of *common sense* (CS) concepts about how the world works. To evaluate introductory physics instruction, the *Force Concept Inventory* (FCI) was developed to detect differences (in student thinking) between CS concepts and Newtonian concepts about motion and its causes [6]. Results from applying the FCI were stunning from the get-go! First, the differences were huge before instruction. Second, the change was small after instruction. Third, results were independent of the instructor’s experience, teaching method and peer evaluation. These results have been replicated thousands of times from high school to Harvard and in 25 different languages. The FCI is by far the most cited reference in the physics education literature, and it is widely used today to evaluate the effectiveness of teaching reforms.

The FCI tells us a lot about human cognition. It is based on a taxonomy of 35 CS concepts in 5 major categories [6]. These concepts are incompatible with Newtonian physics in almost every detail. However, they are common outcomes from everyday experience, and they are quite serviceable for dealing with physical objects. Moreover, central CS concepts in the 5 categories have been clearly articulated and discussed by major intellects of the pre-Newtonian age, including Newton himself before the *Principia* [7]. So CS concepts should be regarded as alternative hypotheses about the physical world that, when clearly formulated, can be tested empirically.

Ability to distinguish between CS concepts and scientific concepts in the FCI or elsewhere is not a matter of intelligence but of experience. It is acquired only by engagement with science itself, usually through academics. These facts suggest that the transition from common sense to scientific thinking is not a replacement of CS concepts with scientific concepts, but rather a realignment of intuition with experience. Science does not replace common sense. Rather, as Modeling Theory aims to show, science is a refinement of common sense differing in respect to:

- objectivity – with explicit rules & conventions for observer-independent inferences,
- precision – in measurement, in description and analysis,
- formalization – for mathematical modeling and analysis of complex systems,
- systematics – coherent, consistent & maximally integrated bodies of knowledge,
- reliability – critically tested & reproducible results,
- skepticism – about unsubstantiated claims.

### III. “What, precisely, is thinking?” — *Einstein*

Research in cognitive science with bearing on Modeling Theory has been reviewed in [3]. Only a few examples can be discussed here.

In a probing study into common sense notions of force [8], Andy diSessa identifies a structure in common sense intuition that he calls Ohm’s p-prim. As he explains,

**Ohm’s p-prim** comprises “an agent that is the locus of an impetus that acts against a resistance to produce a result.”

Evidently this intuitive structure is abstracted from experience in pushing objects. It is an important characterization of the central Force-as-Action metaphor identified by the FCI as a source of misconceptions. It also seems to be fundamental in the intuition of physicists, who often declare “A force is a push or a pull,” without noting its metaphorical root in human action.

More generally, diSessa argues that this structure is fundamental to qualitative reasoning. He notes that the logic of Ohm’s p-prim is

the *qualitative proportion*: more effort  $\Rightarrow$  more result,  
and the *inverse proportion*: more resistance  $\Rightarrow$  less result.

This reasoning structure is often evoked for explanatory purposes in everyday experience.

As disclosed in Ohm’s p-prim, the concept of (causal) **agency** entails a basic

**Causal syntax**: agent  $\rightarrow$  (kind of action)  $\rightarrow$  on patient  $\rightarrow$  result.

DiSessa notes that this intuitive causal syntax can be construed (by metaphorical projection at least) as

**Operator syntax**: agent  $\rightarrow$  (kind of action)  $\rightarrow$  on patient  $\rightarrow$  result,

where the action is on symbols (instead of material objects) to produce other symbols. When the symbols are words, this provides an intuitive base for verb structure expressing the action of mental agents on mental objects. It is a basic aspectual schema for verb structure, recently studied at length in *cognitive grammar* [9].

All this has direct bearing on Kant’s *Critique*. He said Hume woke him from his “dogmatic slumber” with his argument that no amount of empirical data can establish a cause-effect relation between events with certainty. Claiming that Newton’s Laws do establish causality with certainty, Kant argued that it must therefore be known prior to experience (“synthetic a priori”). One can argue instead that the fundamental Laws and Principles of science are discovered as general patterns in experience and simply adopted as postulates in our theories. But Ohm’s p-prim shows that causality is imbedded in the way we think and so it may be a precondition for recognizing causal patterns. In this sense, at least, cognitive science supports Kant’s view.

*Evolutionary psychology* [10] tells us that human brains evolved adaptively to enable navigation to find food and respond to threats. Successful hunting required a number of cognitive abilities: To create mental maps of the environment and plan actions, to design helpful tools, to “read” subtle clues in natural surroundings; and, finally to communicate and cooperate with other humans. The ancient practice of storytelling prepared the human mind for reading written text when it was invented. Research on *narrative comprehension* [11] supports the view that readers of stories construct mental models of the situation, characters and events described.

Research by Philip Johnson-Laird [12] supports the claim that *most human reasoning is inference from mental models*. We can distinguish several types of **model-based reasoning**:

- **Abductive**, to complete or extend a model, often guided by a semantic frame in which the model is embedded.
- **Deductive**, to extract substructure from a model.
- **Inductive**, to match models to experience.
- **Analogical**, to interpret or compare models.
- **Metaphorical**, to infuse structure into a model.
- **Synthesis**, to construct a model, perhaps by analogy or blending other models.
- **Analysis**, to profile or elaborate implicit structure in a model.

*Justification* of model-based reasoning requires translation from mental models to *inference from conceptual models* that can be publicly shared, like the scientific models discussed below.

Modeling Theory (see below) holds that mathematicians and even logicians reason mostly from mental models. Model-based reasoning is more general and powerful than propositional logic, as it integrates multiple representations of information (propositions, maps, diagrams, equations) into a coherently structured mental model. Rules and procedures are central to the formal concept of inference, but they can be understood as prescriptions for operations on mental models as well as on symbols.

Barbara Tversky and collaborators [13] have tested the classical view that *mental imagery is internalized perception*. They compared individual accounts of a visual scene generated from narrative with accounts generated from direct observation and found them *functionally equivalent*. A crucial difference is that perceptions have a fixed point of view, while mental models allow change in point of view. Furthermore, spatial mental models are more schematic and categorical than images, capturing some features of the object but not all and incorporating information about the world that is not purely perceptual. The general conclusion is that *mental models represent states of the world as conceived, not perceived*. To know a thing is to form a mental model of it.

The best fit to data is a *spatial framework model*, where each object has an *egocentric frame* consisting of mental extensions with three body axes.

### **Spatial MENTAL models**

- are *schematic*, representing only some features,
- are *structured*, consisting of *elements and relations*.
- **Elements are typically objects** (or reified things).
- **Object properties are idealized** (points, lines or paths).
- Object models are always *placed in a background* (context or **frame**).
- Individual objects are *modeled separately* so they can move around in the frame.

Details in this list are richly supported by research in “cognitive linguistics” [14], a new approach to linguistics is grounded in the **revolutionary thesis**: *Language does not refer directly to the world, but rather to mental models and components thereof! Words serve to activate, elaborate or modify mental models, as in comprehension of a narrative*.

This thesis rejects all previous versions of semantics, which located the referents of language outside the mind, in favor of **cognitive semantics**, which locates referents inside the mind. I see the evidence supporting cognitive semantics as overwhelming, but it must be admitted that some linguists are not convinced, and many research questions remain. Cognitive

semantics can be regarded as a culmination of Kant's revolution toward an epistemology grounded in science.

From the domain of *cognitive neuroscience*, Stanislas Dehaene reports [16]: "Mathematicians frequently evoke their "intuition" when they are able to quickly and automatically solve a problem, with little introspection into their own insight. His research shows that "automaticity aspect" of mathematical intuition can be studied in the laboratory in reduced paradigms, and that relates to the availability of "core knowledge" associated with evolutionarily ancient and specialized cerebral subsystems." Subsystems involved in basic operations of arithmetic (such as number estimation, comparison, addition and subtraction) have been identified as genetically hardwired. The boundary between hardwired and learned mathematical abilities continues to be a rich area for further research.

The empirical research cited above supports an answer to Einstein's question: *Thinking* is a hardwired human ability to freely create mental models and use them for planning and controlling interactions with the physical world. To deepen this insight and coordinate empirical results, we need a scientific theory, to which we now turn.

#### IV. MODELING THEORY

Though Modeling Theory is proposed as a general theory of Mind embracing all aspects of cognitive science, we limit our attention here to cognition in physics and mathematics. The natural languages give us rich information about the structure of mental models in common sense cognition. Complimentary and reinforcing results come from the language of science, especially mathematics. After spelling out the structure of **scientific models** below, we discuss implications for cognition in physics and mathematics.

Our formulation of Modeling Theory rests on explication of two key concepts "model" and "morphism." We begin with the definition:

A **model** is a **representation of structure** in a given **system**.

A *system* is a set of related *objects*, which may be real or imaginary, physical or mental, simple or composite. The *structure* of a system is a set of relations among its objects. The system itself is called the *referent* of the model. We often identify the model with its *representation* in a concrete inscription of words, symbols or figures (such as graphs, diagrams or sketches). But it must not be forgotten that the inscription is supplemented by a system of (mostly tacit) rules and conventions for encoding model structure.

From my experience as a scientist, I have concluded that **five types of structure suffice** to characterize *any scientific model*. Although my initial analysis was based on physics, I have concluded the classification is sufficient for all other sciences as well. As this seems to be an important empirical fact, a brief description of each type is in order here.

**Universal structures in scientific models** [2, 17]:

- **Systemic structure:** Its representation specifies (a) *composition* of the system (b) *links* among the parts (individual objects), (c) links to *external agents* (objects in the environment). A diagrammatic representation is usually best (with objects represented by nodes and links represented by connecting lines) because it provides a wholistic image of the entire structure. Examples: electric circuit diagrams, organization charts, family trees.
- **Geometric structure:** specifies (a) *configuration* (geometric relations among the parts), (b) *location* (position with respect to a reference frame)

- **Object structure:** *intrinsic properties* of the parts. For example, mass and charge if the objects are material things, or *roles* if the objects are *agents* with complex behaviors. The objects may themselves be systems (such as atoms composed of electrons and nuclei), but their internal structure is not represented in the model, though it may be reflected in the attributed properties.
- **Interaction structure:** properties of the links (typically *causal* interactions). Usually represented as binary relations on object pairs. Examples of interactions: forces (momentum exchange), transport of materials in any form, information exchange.
- **Temporal (event) structure:** *temporal change in the state* of the system. Change in position (motion) is the most fundamental kind of change, as it provides the basic measure of time. Measurement theory specifies how to quantify the properties of a system into property variables. The state of a system is a set of values for its property variables (at a given time). Temporal change can be represented *descriptively* (as in graphs), or *dynamically* (by equations of motion or conservation laws).

Optimal precision in definition and analysis of structure is supplied by **mathematics, the science of structure.**

Both the model and its referent are structured objects, but they need not be distinct. Indeed, the usual notion of a *mathematical model* as a representation in terms of mathematical symbols does not specify any referent, so we say it is an *abstract model*. Of course, it is a perfect representation of itself. This suggests that we regard any structured object as an “abstract model.”

Our definition of “model” above is likewise abstract, because it does not specify the worlds (domains) in which the model and its referent exist as structured objects. To address this issue, Modeling Theory [3] posits three distinct worlds in which structured objects exist:

- World 1: The PHYSICAL WORLD of real things and events, including biological entities.
- World 2: The MENTAL WORLD of **mental models** generated by perception or intuition.
- World 3: The CULTURAL WORLD of human artifacts, including natural languages and mathematics in any form, written or spoken.

This helps us make a crucial distinction between mental models and conceptual models. **Mental models** are private constructions in the mind of an individual (World 2). They can be elevated to **conceptual models** by encoding model structure in symbols (World 3) that activate the individual’s mental model and corresponding mental models in other minds. Thus, communication between individuals involves construction and use of shared *conceptual models*.

Note that a conceptual model establishes an *analogy* between a mental model and its symbolic representation. Mathematical models are symbolic structures, and to understand one is to create a mental model with analogous structure. An *analogy* is defined as a *mapping of structure* from one domain (*source*) to another (*target*) [18]. The mapping is always partial, which means that some structure is not mapped. Science sets up many kinds of analogy between and within the three worlds [3]. Thus, experimental testing or simply interpreting a scientific model (World 3) requires a mapping to a physical system (World 1) that I call a *referential analogy*. *Material analogies* relate structures of different physical objects in World 1 and this

reduces to an *inductive analogy* when the objects are regarded as identical. And there are many more analogies with *computer models* (World 1).

There are other kinds of structure-preserving mappings such as *metaphors*, which Lakoff [15] defines as a projection of structure from one domain into another. I recommend formalizing all such concepts with the technical term **MORPHISM**. In mathematics a *morphism is a structure-preserving mapping*: Thus the terms *homomorphism* (preserves algebraic structure) and *homeomorphism* (preserves topological structure).

Now let us reconsider Kant's trenchant analysis of thinking in physics and mathematics. *Physical intuition* is accorded the same high regard by physicists that mathematicians accord to *mathematical intuition*. To quote unquestionable leaders in each field [2]: Einstein explains,

"The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. . . . The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be voluntarily reproduced and combined. . . ."

Hilbert asserts,

"No more than any other science can mathematics be founded on logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought."

Modeling theory asserts that physical and mathematical intuitions are merely two different ways to relate products of imagination to the external world. **Physical intuition** matches structure in mental models with structure in physical systems. **Mathematical intuition** matches mental structure with symbolic structure. Thus, structure in imagination is common ground for both physical and mathematical intuition.

Kant reasoned in much the same way. He also took the physics and mathematics of his day as given and asked what makes them so special. His analysis is cogent even today, so key points are worth reconsidering. He began by identifying *construction in intuition* as *a means* for acquiring certain geometrical knowledge:

"Thus we think of a triangle as an **object**, in that we are conscious of the combination of the straight lines according to a **rule** by which such an **intuition** can always be **represented**. . . This representation of a universal procedure of imagination in providing an image for a concept, I entitle the **schema of this concept**."

Kant did not stop there. Like any good scientist he anticipated objections to his hypothesis. Specifically, he noted that his intuitive image of a triangle is always a *particular triangle*. How, he asks, can construction of a concept by means of a single figure "express universal validity for all possible intuitions which fall under the same concept?" This is the general epistemological *problem of universality* for the case of Kant's theory of geometrical proof. Kant's notion of geometrical proof is by construction of figures, and he argues that such proofs have universal validity as long as the figures are "determined by certain universal conditions of construction." In other words, construction in intuition is a *rule-governed activity* that makes it possible for geometry to discern "the universal in the particular."

Kant's argument is often dismissed because it led him to conclude that Euclidean geometry is *certain a priori*. But that is a red herring! Because we now know that non-Euclidean

geometry can be associated with the same intuitive construction simply by changing the rules assigned to it. His essential point is that **mathematical inference from intuition is governed by subsumption under rules**. As mathematician Saunders MacLane [19] asserts, “Mathematics is not concerned with reality but with rule.”

## V. RULES AND TOOLS FOR THINKING AND DOING

Science and technology have coevolved with language and mathematics. The evolution is driven by invention of tools with increasing sophistication and power to shape and understand the physical world. The tools of science are of two kinds: instruments for detecting reproducible patterns in the material world, and symbolic systems to represent those patterns for contemplation in the mind.

The detection of patterns in nature began with direct observation using human sensory apparatus. Then the human perceptual range was extended by scientific instruments, such as telescopes and microscopes. Finally, Technology has replaced human sensory detectors with more sensitive instruments, and the data is processed by computers with no role for humans except to interpret the final results; even there the results may be fed to a robot to take action with no human participation at all.

Tool development in the cognitive domain began with the natural languages in spoken and then written form. Considering their *ad hoc* evolution, the coherence, flexibility and subtlety of the natural languages is truly astounding. More deliberate and systematic development of symbolic tools came with the emergence of science and mathematics. The next stage of enhancing human cognitive powers with computer tools is just beginning.

While science is a search for structure, mathematics is the science of structure. Every science develops specialized modeling tools to represent the structure it investigates. Witness the rich system of diagrams that chemists have developed to characterize atomic and molecular structure. Ultimately, though, these diagrams provide grist for mathematical models of greater explanatory power. What accounts for the ubiquitous applicability of mathematics to science? An answer is suggested by considering the coevolution of mathematics and physics from the perspective of modeling theory.

Tools of technology provide an obvious index of progress in human civilization, because their results are so tangible. A more subtle and informative index is the development of language and mathematics, which provide us with **tools to think with!** Though spoken language reaches back more than 150,000 years, written language is barely 5,000 years old, and printed books less than 700. With the invention of calculus by Newton and Leibniz in the seventeenth century, the development of mathematics and physics has accelerated to this day. *Kant put his finger on the source of this stunning revolution: the use of rules to harness the powers of human intuition.*

Precision in science requires precise standards and conventions, in short, precise rules in both empirical and theoretical domains. The coevolution of physics and mathematics has been driven by invention and application of new rules to shape human intuition and model the physical world. The tools of technology from simple hand tools to complex machines were obviously invented. Likewise the tools of mathematics were invented, not discovered; though it may be said that theorems derived from structures built with those tools are discovered.

The vicissitudes of mathematical invention are evident in the motley assortment of mathematical tools used by physicists today, from vectors and matrices to tensors, spinors and differential forms. Far from exhibiting the unity and richness of mathematics, these “tool kits” contribute redundancy, inefficiency and obscurity [21]. A more coherent and powerful system of mathematical tools explicitly designed to integrate algebra and geometry is already well

developed with a huge range of applications. Few physicists and mathematicians know about it, so an introduction to the literature is appropriate here, especially as it supports the present thesis of *mathematics by design!*

Kant himself contributed to the rule-based developments in mathematics. He was the first to formulate the abstract commutative and associative rules for addition (published by his mathematician friend Johann Schultz). Within the next century, Hermann Grassmann and W. K. Clifford provided foundations for integrating geometry, algebra and calculus into a *universal geometric calculus* that is developing with renewed vigor today. A history of *geometric algebra and calculus* is given in [20]. Its implications for the design of mathematical tools to simplify and unify the physics and mathematics curriculum are discussed in [21]. Extension to modeling spacetime, quantum mechanics and gauge theory gravity is given in [22,23].

To the question: “*What is man?*”

Aristotle answered: “*Man is a rational animal.*”

Anthropologists observe: “*Man is a tool-making animal.*”

Modeling Theory suggests: “*Man is a modeling animal!*”     **Homo modelus!**

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