ROTATIONAL DYNAMICS WITH GEOMETRIC ALGEBRA

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Abstract. A new spinor formulation of rotational dynamics is developed. A general theorem is established reducing the theory of the symmetric top to that of the spherical top. The classical problems of Lagrange and Poinsot are treated in detail, along with a modern application to the theory of magnetic resonance.

For treating rotations and rotational dynamics in three dimensions, quaternions are demonstrably more efficient than matrices. Ickes (1970) establishes this fact by counting and comparing the elementary operations needed to compose rotations on a computer. Geometric algebra makes the full power of quaternion algebra readily available by integrating it with conventional vector algebra in a larger coherent system.

The fundamentals of geometric algebra and its application to rotations have been laid out in Hestenes (1983), which is a prerequisite to this paper. To prepare the system for wide applications to rotational dynamics, it is necessary to reformulate classical results of this field within the system. That is the aim of this paper. The classical problems of Lagrange and Poinsot, as well as a fundamental problem in magnetic resonance, are formulated and solved in new ways, as part of a general spinor theory of rotational dynamics.

Besides preserving and improving the quaternion theory of rotations, the spinor theory developed here applies also to quantum mechanics and has a straightforward relativistic generalization (Hestenes 1974). But we will not be concerned with those matters here.

1. Rotational Equations of Motion

The rotational motion of a rigid body is determined by a pair of coupled first order differential equations, a kinematical and a dynamical equation. Rotational kinematics has been given a spinor formulation in Hestenes (1983), so we simply write down the results we need.

The *attitude* of a rigid body is usually represented by a *body frame* $\{\mathbf{e}_k, k = 1, 2, 3\}$. This is a righthanded orthonormal frame of *directions* (unit vectors) 'rigidly attached' to the body. Let $\{\boldsymbol{\sigma}_k, k = 1, 2, 3\}$ be a *reference frame* consisting of directions which are constant in every inertial system. The body frame is related to the reference frame

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by

$$\mathbf{e}_k = R^\dagger \boldsymbol{\sigma}_k R, \tag{1.1}$$

where R is a unimodular spinor (quaternion). Hence, the attitude of the rigid body can be described by the *attitude spinor R*. The *kinematical equation of motion* is then

$$\dot{R} = \tfrac{1}{2} R i \boldsymbol{\omega}, \tag{1.2}$$

where the dot indicates time derivative and $\boldsymbol{\omega}$ is the *angular velocity* of the rigid body. It follows that

$$\dot{\mathbf{e}}_k = \boldsymbol{\omega} \times \mathbf{e}_k, \tag{1.3}$$

but we hardly need these equations when we have the spinor equation (1.2).

The *dynamical equation* for rotational motion of a rigid body is *Euler's equation*

$$\dot{\mathbf{L}} = \boldsymbol{\Gamma}, \tag{1.4}$$

where $\boldsymbol{\Gamma}$ is the applied *torque*, and the *angular momentum* $\mathbf{L}$ is related to the angular velocity by the 'constitutive equation'

$$\mathbf{L} = \mathscr{I}(\boldsymbol{\omega}). \tag{1.5}$$

The *inertia tensor* $\mathscr{I}$ is a positive definite symmetric linear function. We take the $\mathbf{e}_k$ to be *principle vectors* of the inertia tensor, so

$$\mathscr{I}(\mathbf{e}_k) = I_k \mathbf{e}_k, \tag{1.6}$$

where the *principle values* I_k are positive scalar constants.

To sum up, we describe rotational motion by the dynamical equation (1.4) and the kinematical equation (1.2). These equations are coupled by the constitutive equations (1.5) and (1.6) and the fact that, in general, the torque $\boldsymbol{\Gamma}$ is a function of the spinor R.

2. Reduction of the Symmetric Top

For a symmetric top with *symmetry axis* $\mathbf{e}_3$ we have $I_1 = I_2 = I$ and $I \neq I_3$, so the constitutive equation (1.5) can be put in the explicit form

$$\mathbf{L} = I\boldsymbol{\omega} + (I_3 - I)\boldsymbol{\omega}\cdot\mathbf{e}_3\mathbf{e}_3. \tag{2.1}$$

The inverse of this linear function is

$$\boldsymbol{\omega} = I^{-1}\mathbf{L} + (I_3^{-1} - I^{-1})\mathbf{L}\cdot\mathbf{e}_3\mathbf{e}_3. \tag{2.2}$$

Inserting this into the kinematical equation (1.2), we obtain

$$\dot{R} = \tfrac{1}{2} R i \boldsymbol{\omega}_1 + \tfrac{1}{2} i \boldsymbol{\omega}_2 R, \tag{2.3a}$$

where

$$\boldsymbol{\omega}_1 = I^{-1}\mathbf{L} \tag{2.3b}$$

and

$$\boldsymbol{\omega}_2 = (I_3^{-1} - I^{-1})\mathbf{L}\cdot\mathbf{e}_3\boldsymbol{\sigma}_3 . \tag{2.3c}$$

When $\boldsymbol{\Gamma} = 0$, both $\mathbf{L}$ and $\mathbf{L}\cdot\mathbf{e}_3$ are constants of motion, so (2.3a) integrates immediately to

$$R = e^{(1/2)i\omega_2 t} R_0 e^{(1/2)i\omega_1 t}, \tag{2.4}$$

where the constant spinor R_0 determines the initial attitude of the top. The motion of the symmetry axis $\mathbf{e}_3$ is therefore

$$\mathbf{e}_3(t) = e^{-(1/2)i\omega_1 t}\mathbf{e}_3(0)\, e^{(1/2)i\omega_1 t}. \tag{2.5}$$

This tells us that the symmetry axis precesses about $\mathbf{L}$ with a constant angular velocity $\boldsymbol{\omega}_1$. In accordance with (2.3), the first factor in (2.4) tells us that the body spins about the symmetry axis with an angular speed $\omega_2 = \boldsymbol{\omega}_2\cdot\boldsymbol{\sigma}_3$ while the axis precesses. This motion is sometimes called the *Eulerian Free Precession*. The motion appears as spinning with respect to an inertial frame, but it can be interpreted as precession with respect to the body frame. The resultant angular velocity for the composite motion described by (2.4) is

$$\begin{aligned}\boldsymbol{\omega} &= \frac{2}{i}R^\dagger\dot{R} = \boldsymbol{\omega}_1 + R^\dagger\boldsymbol{\omega}_2 R = \boldsymbol{\omega}_1 + \omega_2\mathbf{e}_3 \\ &= e^{-(1/2)i\omega_1 t}(\boldsymbol{\omega}_1 + \omega_2\mathbf{e}_3(0))\, e^{(1/2)i\omega_1 t}.\end{aligned} \tag{2.6}$$

This tells us that $\boldsymbol{\omega}$ precesses along with $\mathbf{e}_3$ about $\mathbf{L} = I\boldsymbol{\omega}_1$, so the three vectors $\mathbf{L}$, $\boldsymbol{\omega}$, and $\mathbf{e}_3$ remain at all times in a common precessing plane at fixed angles relative to one another.

For a top subject to an arbitrary torque, Equations (2.3a, b, c) remain true, and they allow us to write R in the form

$$R = e^{(1/2)i\sigma_3\psi}\, U, \tag{2.7}$$

where

$$\dot{U} = U\frac{i\mathbf{L}}{2I} \tag{2.8}$$

and

$$\psi(t) = \int_0^t \mathrm{d}t(I_3^{-1} - I^{-1})\mathbf{L}\cdot\mathbf{e}_3 . \tag{2.9}$$

Note that

$$\mathbf{e}_3 = R^\dagger\boldsymbol{\sigma}_3 R = U^\dagger\boldsymbol{\sigma}_3 U. \tag{2.10}$$

Therefore, if we first use (2.8) to get U, then we can determine ψ by the quadrature (2.9) and get R from (2.7). Thus, we have reduced the general kinematical equation (2.3a) to the simpler equation (2.8). Note that the integral (2.9) holds even when there are time variations of I_3 and I, so it may be useful for analyzing the *Chandler Wobble.*

Equation (2.8) is the kinematical equation for a *spherical top* with angular momentum $\mathbf{L}$ and moment of Inertia I. For the most important case of a resultant force $\mathbf{F}$ acting at a point $\mathbf{r} = r\mathbf{e}_3$ on the symmetry axis, the dynamical equation is

$$\dot{\mathbf{L}} = r\mathbf{e}_3 \times \mathbf{F}. \tag{2.11}$$

According to (2.10), this equation couples only to the spinor U for the spherical top. Thus, *we have reduced the equations of motion for a symmetric top to those for a spherical top.* This *reduction theorem* has been noted in a somewhat less general form by Whittaker (1944, p. 159).

3. The Spherical Top

We proceed now to study the motion of a spherical top with the knowledge that the results apply also to the symmetric top, since the reduction theorem tells us that the two kinds of top differ only in the rotation rate about the symmetry axis.

The spherical top has a spherically symmetric inertia tensor with a single principle value I. Hence the angular momentum $\mathbf{L}$ is related to the angular velocity $\boldsymbol{\omega}$ of the top by

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{3.1}$$

Therefore, the dynamical equation (2.11) can be put in the form

$$\dot{\boldsymbol{\omega}} = \mathbf{e} \times \mathbf{G}, \tag{3.2}$$

where

$$\mathbf{G} = I^{-1} r\mathbf{F} \tag{3.3}$$

and

$$\mathbf{e} = U^{\dagger}\boldsymbol{\sigma}_3 U. \tag{3.4}$$

The corresponding kinematical equation is, of course,

$$\dot{U} = \tfrac{1}{2} U i\boldsymbol{\omega}. \tag{3.5}$$

Our problem now is, for given $\mathbf{G}$, to solve the two Equations (3.2) and (3.5) coupled by (3.4).

Our problem can be simplified somewhat by combining the coupled first order differential equations into a single second order differential equation. Thus,

$$\begin{aligned}\ddot{U} &= \tfrac{1}{2} U(i\dot{\boldsymbol{\omega}} - \tfrac{1}{2}\omega^2) = \tfrac{1}{2} U(i\mathbf{e} \times \mathbf{G} - \tfrac{1}{2}\omega^2)\\ &= \tfrac{1}{2} U(\mathbf{e}\mathbf{G} - \mathbf{e}\cdot\mathbf{G} - \tfrac{1}{2}\omega^2).\end{aligned}$$

Thus, we obtain the spinor equation of motion

$$\ddot{U} = \tfrac{1}{2}\boldsymbol{\sigma}_3 U\mathbf{G} - \tfrac{1}{2}\lambda U \tag{3.6}$$

where

$$\lambda = \tfrac{1}{2}\omega^2 + \mathbf{e}\cdot\mathbf{G} = 2|\dot{U}|^2 + \langle U^\dagger \boldsymbol{\sigma}_3 U\mathbf{G}\rangle. \tag{3.7}$$

Although, the rotational dynamics is now in Equation (3.6) subject to (3.7), we are still not done with the original dynamical equation (3.2), because we can use it to derive constants of motion.

Dotting (3.2) with $\mathbf{e}$, we find the constant of motion

$$\mathbf{e}\cdot\boldsymbol{\omega} = \langle \mathbf{e}\boldsymbol{\omega}\rangle = -2\langle i\boldsymbol{\sigma}_3 \dot{U} U^\dagger\rangle. \tag{3.8}$$

If the direction of the applied force $\hat{\mathbf{G}}$ is a constant, then (3.2) gives us another constant of motion

$$\hat{\mathbf{G}}\cdot\boldsymbol{\omega} = -2\langle i\hat{\mathbf{G}} U^\dagger \dot{U}\rangle. \tag{3.9}$$

If the magnitude of $\mathbf{G}$ is also constant, then (3.2) gives us the *energy* constant of motion

$$E = \tfrac{1}{2}\omega^2 - \mathbf{e}\cdot\mathbf{G} = 2|\dot{U}|^2 - \langle U^\dagger \boldsymbol{\sigma}_3 U\mathbf{G}\rangle. \tag{3.10}$$

Our present analysis will be restricted to the important case of constant $\mathbf{G}$, the so-called *Lagrange problem*. Then we can replace (3.7) by the simpler expression

$$\lambda = E + 2\mathbf{e}\cdot\mathbf{G} = E + 2\langle U^\dagger \boldsymbol{\sigma}_3 U\mathbf{G}\rangle. \tag{3.11}$$

This reduces the formidable nonlinearity of our equation of motion (3.6).

For the Lagrange Problem, the three constants of motion (3.8), (3.9), and (3.10) amount to integrating the dynamical equation (3.2) or, equivalently, to integrating the spinor equation once. The classical approach to the Lagrange problem proceeds by parametrizing the rotation with Euler angles. We can do this by writing

$$U = e^{(1/2)i\sigma_3\psi}\, e^{(1/2)i\sigma_1\theta}\, e^{(1/2)i\sigma_3\phi}. \tag{3.12}$$

Then,

$$\boldsymbol{\omega} = -2iU^\dagger\dot{U} = \dot{\phi}\boldsymbol{\sigma}_3 + \dot{\theta}\mathbf{n} + \dot{\psi}\mathbf{e}, \tag{3.13}$$

where

$$\mathbf{n} = \frac{\boldsymbol{\sigma}_3 \times \mathbf{e}}{|\boldsymbol{\sigma}_3 \times \mathbf{e}|} = \boldsymbol{\sigma}_1 e^{i\sigma_3\phi}. \tag{3.14}$$

We are free to specify the vector $\boldsymbol{\sigma}_3$ by writing $\boldsymbol{\sigma}_3 = -\hat{\mathbf{G}}$. The minus sign is convenient for gyroscopic problems where the direction $\hat{\mathbf{G}}$ of the applied force is opposite to the upward vertical direction $\boldsymbol{\sigma}_3$. Now by inserting (3.13) into our expressions for the three constants of motion we obtain

$$\mathbf{e}\cdot\boldsymbol{\omega} = \dot{\phi}\cos\theta + \dot{\psi}, \tag{3.15a}$$

$$\boldsymbol{\sigma}_3 \cdot \boldsymbol{\omega} = \dot{\phi} + \dot{\psi} \cos \theta, \tag{3.15b}$$

$$E = \tfrac{1}{2}(\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \cos \theta) + G \cos \theta. \tag{3.15c}$$

These three equations can be integrated for the Euler angles, and substitution of the results into (3.12) completes the solution. To extend this solution to a symmetric top, the reduction theorem tells us that we need only shift ψ by the amount

$$\Delta\psi = (II_3^{-1} - 1)\boldsymbol{\omega} \cdot \mathbf{e}_3 t \tag{3.16}$$

specified by (2.9).

Analytical solutions of (3.15a, b, c) in terms of elliptic functions are discussed by many authors, such as Synge and Griffith (1949) and Whittaker (1944). The trouble with these solutions is that they are difficult to interpret. For this reason, we take a different approach, solving the spinor equation (3.6) directly with the help of the constants of motion as side conditions. We first look at the special case of steady precession and then find a practical approximate solution for deviations from steady precession which is easy to interpret and apply. In the next section we return to the question of determining the best form for the exact solution.

3.1. Steady precession

To find a solution of (3.6) we first note that the terms on the right side of the equation must be cancelled by $\ddot{U}$ on the left, so the factors $\boldsymbol{\sigma}_3$ and $\mathbf{G}$ must be produced by differentiation. This suggests that we try a solution of the form

$$U = U_1 U_2,$$

where

$$\dot{U}_1 = \tfrac{1}{2} i\boldsymbol{\omega}_1 U_1 \quad \text{and} \quad \dot{U}_2 = \tfrac{1}{2} U_2 i\boldsymbol{\omega}_2.$$

Then

$$\dot{U} = \tfrac{1}{2} i\boldsymbol{\omega}_1 U + \tfrac{1}{2} U i\boldsymbol{\omega}_2, \tag{3.17}$$

and

$$\ddot{U} = -\tfrac{1}{2}\boldsymbol{\omega}_1 U \boldsymbol{\omega}_2 - \tfrac{1}{4}(\omega_1^2 + \omega_2^2)U + \tfrac{1}{2} i(\dot{\boldsymbol{\omega}}_1 U + U\dot{\boldsymbol{\omega}}_2). \tag{3.18}$$

Comparing this with (3.6), we see that a solution with constant $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ obtains if

$$\hat{\boldsymbol{\omega}}_1 = \boldsymbol{\sigma}_3, \quad \hat{\boldsymbol{\omega}}_2 = \hat{\mathbf{G}}, \tag{3.19a}$$

$$\omega_1 \omega_2 = -G, \tag{3.19b}$$

and

$$\tfrac{1}{2}(\omega_1^2 + \omega_2^2) = \lambda, \tag{3.19c}$$

where $\lambda = E + 2\mathbf{G} \cdot \mathbf{e}$ and $\mathbf{G} \cdot \mathbf{e}$ must be constant. The last two of these equations can be solved for ω_1 and ω_2, but a different expression for ω_1 and ω_2 will be more useful.

From (3.17) and (3.19a) we obtain

$$\boldsymbol{\omega} = \frac{2}{i} U^\dagger \dot{U} = U^\dagger \boldsymbol{\omega}_1 U + \boldsymbol{\omega}_2 = \omega_1 \mathbf{e} + \omega_2 \hat{\mathbf{G}},$$

whence

$$\boldsymbol{\omega}\cdot\mathbf{e} = \omega_1 + \omega_2 \hat{\mathbf{G}}\cdot\mathbf{e}. \tag{3.20}$$

Solving (3.20) and (3.19b), we obtain two pairs of solutions

$$\omega_1 = \tfrac{1}{2}\{\boldsymbol{\omega}\cdot\mathbf{e} \mp [(\boldsymbol{\omega}\cdot\mathbf{e})^2 + 4\mathbf{G}\cdot\mathbf{e}]^{1/2}\}, \tag{3.21}$$

$$\omega_2 = (2\hat{\mathbf{G}}\cdot\mathbf{e})^{-1}\{\boldsymbol{\omega}\cdot\mathbf{e} \pm [(\boldsymbol{\omega}\cdot\mathbf{e})^2 + 4\mathbf{G}\cdot\mathbf{e}]^{1/2}\}. \tag{3.22}$$

It is readily verified that these expressions satisfy (3.19c) identically, so there is no additional information in that constraint.

Having established that the conditions for a solution have been satisfied, we can write the solution explicitly:

$$U = e^{(1/2)i\omega_1 t} U_0 e^{(1/2)i\omega_2 t}. \tag{3.23}$$

Like (2.4), this describes a body spinning about its axis $\mathbf{e}$ with an angular speed ω_1 while it precesses *steadily* with angular velocity $\boldsymbol{\omega}_2$.

For a rapidly spinning body, the kinetic energy is much greater than the potential energy. Therefore, $(\boldsymbol{\omega}\cdot\mathbf{e})^2 \gg |4\mathbf{G}\cdot\mathbf{e}|$, and, to a good approximation, Equation (3.22) reduces to the *fast top* solution

$$\omega_2 = \frac{\boldsymbol{\omega}\cdot\mathbf{e}}{\hat{\mathbf{G}}\cdot\mathbf{e}} = -\frac{G}{\omega_1}$$

and the *slow top* solution

$$\omega_2 = -\frac{G}{\boldsymbol{\omega}\cdot\mathbf{e}} = -\frac{G}{\omega_1}.$$

The fast top solution describes an upright top if $\hat{\mathbf{G}}\cdot\mathbf{e} < 0$ or a *hanging top* (or gyroscopic pendulum) if $\hat{\mathbf{G}}\cdot\mathbf{e} > 0$.

3.2. Deviations from Steady Precession

The solution for steady precession which we have just examined is an exact solution of the spinor equation of motion, but it is a special solution. However, for any total energy E there is always a solution with steady precession. It differs from other solutions with the same total energy in having the kinetic and potential energies as separate constants of motion. Therefore, we can describe any solution in terms of its deviation from steady precession. Accordingly, we write the solution in the form

$$U = U_1 D U_2 \tag{3.24}$$

where

$$U_1 = e^{(1/2)i\omega_1 t} \quad \text{with} \quad \boldsymbol{\omega}_1 = \omega_1 \mathbf{e}_0, \tag{3.25a}$$

$$U_2 = e^{(1/2)i\omega_2 t} \quad \text{with} \quad \boldsymbol{\omega}_2 = \omega_2 \hat{\mathbf{G}}, \tag{3.25b}$$

and ω_1 and ω_2 are constants determined by

$$\omega_1 \omega_2 = -G, \tag{3.25c}$$

$$\tfrac{1}{2}(\omega_1^2 + \omega_2^2) = E + 2\mathbf{G}\cdot\mathbf{e}_0. \tag{3.25d}$$

The spinor D in (3.24) describes the deviation from steady precession. To obtain a differential equation for D, we substitute (3.24) into the equation of motion

$$\ddot{U} = \tfrac{1}{2}[e_0 U\mathbf{G} - (E + 2\mathbf{G}\cdot\mathbf{e})U] \tag{3.26}$$

and use (3.25a, b, c, d). Thus, we obtain

$$\ddot{D} + i\omega_1 \dot{D} + \dot{D} i\omega_2 + \mathbf{G}\cdot(\mathbf{e} - \mathbf{e}_0)D = 0, \tag{3.27}$$

where $\mathbf{e}_0$ is a constant vector and

$$\mathbf{e} = U^\dagger \mathbf{e}_0 U = U_2^\dagger \mathbf{e}_1 U_2 \tag{3.28}$$

with

$$\mathbf{e}_1 = D^\dagger \mathbf{e}_0 D. \tag{3.29}$$

From (3.25b) we have

$$\mathbf{G}\cdot\mathbf{e} = \mathbf{G}\cdot\mathbf{e}_1 = \langle \mathbf{G} D^\dagger e_0 D \rangle. \tag{3.30}$$

Hence the last term (3.27) is a function of D only.

To study small deviations from steady precession, we write

$$D = e^{(1/2)i\boldsymbol{\varepsilon}} \approx 1 + \tfrac{1}{2} i\boldsymbol{\varepsilon}. \tag{3.31}$$

Substituting this into (3.27) and separating scalar and bivector parts we obtain, to first order in $\boldsymbol{\varepsilon}$, the two equations

$$\dot{\boldsymbol{\varepsilon}}\cdot\boldsymbol{\omega}_+ - 2\boldsymbol{\varepsilon}\cdot(\mathbf{e}_0 \times \mathbf{G}) = 0 \tag{3.32}$$

and

$$\ddot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}} \times \boldsymbol{\omega}_- = 0 \tag{3.33}$$

where

$$\boldsymbol{\omega}_\pm = \boldsymbol{\omega}_1 \pm \boldsymbol{\omega}_2 = \omega_1 \mathbf{e}_0 \pm \omega_2 \hat{\mathbf{G}}. \tag{3.34}$$

Equation (3.33) integrates immediately to

$$\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\omega}_- \times \boldsymbol{\varepsilon} \tag{3.35}$$

where the integration constant has been set to zero to satisfy (3.32). The consistency of (3.35) with (3.32) is easily proved after using (3.25) and (3.34) to establish

$$\boldsymbol{\omega}_- \times \boldsymbol{\omega}_+ = 2\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 = -2\mathbf{e}_0 \times \mathbf{G}. \tag{3.36}$$

The solution to (3.35) is the rotating vector

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 e^{i\boldsymbol{\omega}_- t} = \boldsymbol{\varepsilon}_0 e^{i(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)t}, \tag{3.37}$$

where $\boldsymbol{\varepsilon}_0$ is constant vector orthogonal to $\boldsymbol{\omega}_-$. An additive constant vector parallel to $\boldsymbol{\omega}_-$ has been omitted from (3.37), because its only effect would be to change the initial conditions which are already taken care of in specifying the vector $\mathbf{e}_0$.

We now have a complete solution to the equation of motion, and we can write the attitude spinor U in the form

$$U = e^{(1/2)i\boldsymbol{\omega}_1 t}\, e^{(1/2)i\boldsymbol{\varepsilon}(t)}\, e^{(1/2)i\boldsymbol{\omega}_2 t}, \tag{3.38}$$

where $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(t)$ is the rotating vector given by (3.37). This shows explicitly the time dependence of the rotational motion and its decomposition into three simpler motions. As noted before, the first factor in (3.38) describes a rotation of the body about its symmetry axis, while the third factor describes a steady precession. The second factor determines *nutation.* To visualize the motion, we consider the orbit $\mathbf{e} = \mathbf{e}(t)$ of the symmetry axis on the unit sphere.

To first order, substitution of (3.31) into (3.29) gives us

$$\mathbf{e}_1 = \mathbf{e}_0 + \boldsymbol{\varepsilon} \times \mathbf{e}_0 = \mathbf{e}_0 + (\boldsymbol{\varepsilon}_0 e^{i\boldsymbol{\omega}_- t}) \times \mathbf{e}_0. \tag{3.39}$$

Note that the term $\boldsymbol{\varepsilon} \times \mathbf{e}_0$ is a linear function which projects $\boldsymbol{\varepsilon}$ onto a plane with normal $\mathbf{e}_0$ and rotates it through a right angle. Therefore, it projects the circle $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(t)$ into an ellipse with its major axis in the plane containing $\mathbf{e}_0$ and the normal $\boldsymbol{\omega}_- = \omega_1 \mathbf{e}_0 + \omega_2 \hat{\mathbf{G}}$ to the circle. Thus, (3.39) describes an ellipse $\mathbf{e}_1 = \mathbf{e}_1(t)$ centered at $\mathbf{e}_0$ and lying in the tangent plane to the unit sphere, as shown in Figure 1. The eccentricity of the ellipse

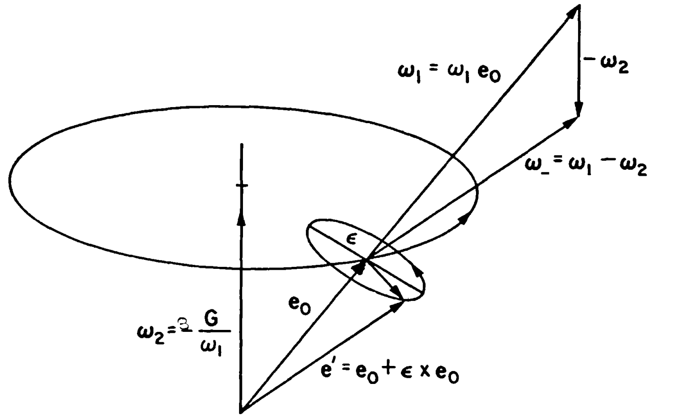

Fig. 1.

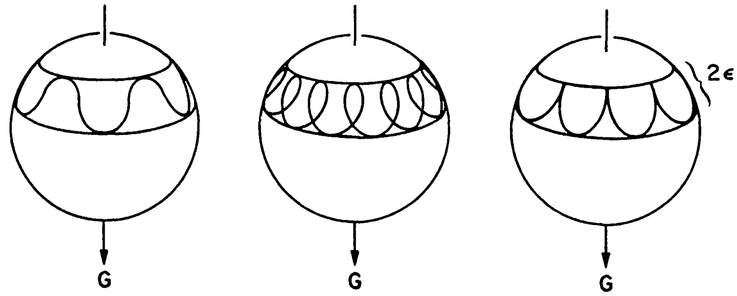

Fig. 2.

depends on the angle between e_0 and $\boldsymbol{\omega}_- = \omega_1 \mathbf{e}_0 + \omega_2 \hat{\mathbf{G}}$. For a slow top $\omega_1 \gg \omega_2$, so the ellipse is nearly circular.

The orbit $\mathbf{e} = \mathbf{e}(t)$ of the symmetry axis is the composite of the elliptical motion (3.39) and the steady precession, as described by

$$\mathbf{e}(t) = R_2^\dagger \mathbf{e}_1 R_2 = e^{-(1/2)i\omega_2 t} \mathbf{e}_1(t) e^{(1/2)i\omega_2 t}. \tag{3.40}$$

The resulting curve oscillates with angular speed

$$\tfrac{1}{2}\omega_- = \left[\tfrac{1}{2}(E + \mathbf{e}_0 \cdot \mathbf{G})\right]^{1/2} \tag{3.41}$$

between two circles on the unit sphere with angular separation $2\varepsilon = 2|\boldsymbol{\varepsilon}|$, as indicated in Figure 2. We use the term *nutation* to designate the elliptical oscillation about steady precession, though the term ordinarily refers only to the vertical 'nodding' component of this oscillation.

To determine the qualitative features of the orbit $\mathbf{e} = \mathbf{e}(t)$, we look at the velocity

$$\dot{\mathbf{e}} = R_2^\dagger [\mathbf{e}_0 \times (\boldsymbol{\omega}_- \times \boldsymbol{\varepsilon} + \boldsymbol{\omega}_2)] R_2. \tag{3.42}$$

The nutation velocity $\mathbf{e}_0 \times (\boldsymbol{\omega}_- \times \boldsymbol{\varepsilon})$ is exactly opposite to the precession velocity $\mathbf{e}_0 \times \boldsymbol{\omega}_2$ only when the orbit is tangent to the upper bounding circle in Figure 2. Therefore, $|\dot{\mathbf{e}}|$ has its minimum values at such points, and $\dot{\mathbf{e}} = 0$ only on an orbit for which

$$\varepsilon |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| = |\mathbf{e}_0 \times \boldsymbol{\omega}_2|. \tag{3.43}$$

This is the condition for the *cuspidal orbit* in Figure 2. A *looping* orbit occurs when $\varepsilon|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| > |\mathbf{e}_0 \times \boldsymbol{\omega}_2|$, and a smooth orbit without loops occurs when $\varepsilon|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| < < |\mathbf{e}_0 \times \boldsymbol{\omega}_2|$. A cuspidal orbit can be achieved in practice by releasing the axis of a spinning top from an initial position at rest. Therefore, the two other kinds of orbits can be achieved with an initial impetus following or opposing the direction of precessional motion.

4. Exact Solution of the Lagrange Problem

With $\mathbf{G} = G\boldsymbol{\sigma}_3$, the equation of motion (3.6) can be put in the form

$$\ddot{U} + \tfrac{1}{2}(E + 2\mathbf{e}\cdot\mathbf{G})U - \tfrac{1}{2}G\boldsymbol{\sigma}_3 U \boldsymbol{\sigma}_3 = 0. \tag{4.1}$$

The factor $\boldsymbol{\sigma}_3 U \boldsymbol{\sigma}_3$ suggests that we might simplify the equation by writing U in the form

$$U = \alpha - i\boldsymbol{\sigma}_2 \beta, \tag{4.2}$$

where α and β are quaternions which commute with $\boldsymbol{\sigma}_3$; for then

$$\boldsymbol{\sigma}_3 U \boldsymbol{\sigma}_3 = \alpha + i\boldsymbol{\sigma}_2 \beta. \tag{4.3}$$

The constant bivector $-i\boldsymbol{\sigma}_2$ has been chosen so α and β are exactly the classical *Cayley–Klein* parameters (Whittaker, 1944, p. 12). Indeed, by Equating (4.2) with (3.12), we get the standard expressions for the Cayley–Klein parameters in terms of Euler angles:

$$\alpha = \cos \tfrac{1}{2}\theta \, e^{(1/2)i\boldsymbol{\sigma}_3(\phi + \psi)}, \tag{4.4a}$$

$$\beta = i\boldsymbol{\sigma}_3 \sin \tfrac{1}{2}\theta \, e^{(1/2)i\boldsymbol{\sigma}_3(\phi - \psi)}, \tag{4.4b}$$

But for us it is essential to interpret the unit imaginary in the standard expressions as the bivector $i\boldsymbol{\sigma}_3$. Anyway, we will not use the Euler angles because we shall see that the Cayley–Klein parameters are more appropriate in the Lagrange problem.

To decompose (4.1) into equations for the Cayley–Klein parameters, first note that $\boldsymbol{\sigma}_2 \alpha = \alpha^\dagger \boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_2 \beta = \beta^\dagger \boldsymbol{\sigma}_2$, so

$$\boldsymbol{\sigma}_3 \mathbf{e} = \boldsymbol{\sigma}_3 U^\dagger \boldsymbol{\sigma}_3 U = |\alpha|^2 - |\beta|^2 - 2i\boldsymbol{\sigma}_2 \alpha\beta.$$

Also,

$$U^\dagger U = |\alpha|^2 + |\beta|^2 = 1. \tag{4.5}$$

Hence

$$\boldsymbol{\sigma}_3 \cdot \mathbf{e} = 2|\alpha|^2 - 1 = 1 - 2|\beta|^2. \tag{4.6}$$

Using this along with (4.2) and (4.3), we separate (4.1) into a pair of independent equations for α and β:

$$\ddot{\alpha} + [\tfrac{1}{2}(E - 3G) + 2G|\alpha|^2]\alpha = 0, \tag{4.7}$$

$$\ddot{\beta} + [\tfrac{1}{2}(E + 3G) - 2G|\beta|^2]\beta = 0. \tag{4.8}$$

Of course, these are equivalent equations since G is simply a parameter.

The great advantage of these equations for α and β is that they involve formally complex functions, so all the resources of complex function theory can be brought to bear on their solution. Indeed, the solutions for α and β in terms of elliptic functions are given by Whittaker (1944, p. 161). However, Whittaker merely notices that the expressions for α and β have a particularly simple form after solving the Lagrange problem in terms of Euler angles. We should expect that the Lagrange problem can be more easily solved with our Equation (4.7) for the Euler parameters. Indeed, the solution to (4.7) ought to have a preferred place in elliptic function theory. But that is a matter that deserves more analysis than we can provide here.

The three constants of motion can be used to get first integrals of (4.7) and (4.8). We find that α and β are related to the constants of motion by

$$\alpha^\dagger\dot{\alpha} + \beta^\dagger\dot{\beta} = \tfrac{1}{2}\boldsymbol{i}\boldsymbol{\sigma}_3\boldsymbol{\omega}\cdot\boldsymbol{\sigma}_3 \tag{4.9}$$

$$\alpha^\dagger\dot{\alpha} + \beta\dot{\beta}^\dagger = \tfrac{1}{2}\boldsymbol{i}\boldsymbol{\sigma}_3\boldsymbol{\omega}\cdot\mathbf{e}_3 \tag{4.10}$$

$$\begin{aligned}|\dot{\alpha}|^2 + |\dot{\beta}|^2 &= \tfrac{1}{2}(E - G) + G|\alpha|^2 \\ &= \tfrac{1}{2}(E + G) - G|\beta|^2 .\end{aligned} \tag{4.11}$$

And it should be noted that the scalar parts of (4.9) and (4.11) vanish because

$$\langle \alpha^\dagger\dot{\alpha} \rangle = \langle \alpha\dot{\alpha}^\dagger \rangle = -\langle \beta^\dagger\dot{\beta} \rangle = -\langle \beta\dot{\beta}^\dagger \rangle . \tag{4.12}$$

Separating α from β in (4.9) and (4.10) we obtain

$$\alpha^\dagger\dot{\alpha} - \alpha\dot{\alpha}^\dagger = \tfrac{1}{2}\boldsymbol{i}\boldsymbol{\sigma}_3\boldsymbol{\omega}\cdot(\boldsymbol{\sigma}_3 + \mathbf{e}) \tag{4.13}$$

$$\beta^\dagger\dot{\beta} - \beta\dot{\beta}^\dagger = \tfrac{1}{2}\boldsymbol{i}\boldsymbol{\sigma}_3\boldsymbol{\omega}\cdot(\boldsymbol{\sigma}_3 - \mathbf{e}) \tag{4.14}$$

Also, from (4.9) and (4.13) we obtain

$$|\alpha|^2|\dot{\alpha}|^2 - |\beta|^2|\dot{\beta}|^2 = \tfrac{1}{4}\boldsymbol{\omega}\cdot\boldsymbol{\sigma}_3\boldsymbol{\omega}\cdot\mathbf{e}_3 \tag{4.15}$$

Using (4.11) to eliminate $|\dot{\beta}|^2$ we get

$$|\dot{\alpha}|^2 - (1 - |\alpha|^2)(\tfrac{1}{2}(E - G) + G|\alpha|^2) = \tfrac{1}{4}\boldsymbol{\omega}\cdot\boldsymbol{\sigma}_3\boldsymbol{\omega}\cdot\mathbf{e}_3 . \tag{4.16}$$

And from (4.13) we can get

$$\alpha|\dot{\alpha}|^2 = \alpha^\dagger\dot{\alpha}^2 - \tfrac{1}{2}\boldsymbol{i}\boldsymbol{\sigma}_3\boldsymbol{\omega}\cdot(\boldsymbol{\sigma}_3 + \mathbf{e})\dot{\alpha}. \tag{4.17}$$

Eliminating $|\dot{\alpha}|^2$ between (4.16) and (4.17) we obtain a quadratic equation for $\dot{\alpha}$ which can be solved so we can integrate once more to get α. However, some finesse will be required to get a suitable expression for the resulting integral in terms of elliptic functions.

Whittaker gives the solution in terms of *Weierstrass elliptic functions*, but *Jacobi elliptic functions* would probably be more appropriate, especially since they are better for numerical computations. This is apparent in the special case of a pendulum oscillating in the $\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1$-plane, for then α and β are scalar valued functions of time, so (4.16) can be integrated directly. When $E < G$, the solution is

$$U = \mathrm{d}n(G^{1/2}t) - \boldsymbol{i}\boldsymbol{\sigma}_2 k\,\mathrm{sn}(G^{1/2}t), \tag{4.18}$$

where $k = \sin\frac{1}{2}\theta_0$ is the modulus of the elliptic functions and θ_0 is the angle of maximum deflection. When $E > G$, the solution is

$$U = \mathrm{cn}\left(\frac{G^{1/2}t}{k}\right) - \boldsymbol{i}\boldsymbol{\sigma}_2\,\mathrm{sn}\left(\frac{G^{1/2}t}{k}\right) \tag{4.19}$$

where $k^2 = 2G(E + G)^{-1}$. This solution describes a pendulum with enough kinetic energy to rotate continuously in one direction.

5. Poinsot's Problem

Our aim now is to show how geometric algebra can be used to solve *Poinsot's Problem*, that is, to find an analytic description for the motion of a freely rotating rigid body with an asymmetric inertia tensor. By combining the dynamical equation $\dot{\mathbf{L}} = 0$ with the kinematic equations $\dot{\mathbf{e}}_k = \boldsymbol{\omega} \times \mathbf{e}_k$, we get *Euler's equations* (for $k = 1, 2, 3$)

$$I_k \dot{\omega}_k + (\boldsymbol{\omega} \times \mathbf{L}) \cdot \mathbf{e}_k = 0. \tag{5.1}$$

The solution of these equations for the angular velocity components

$$\omega_k = \boldsymbol{\omega} \cdot \mathbf{e}_k = I_k^{-1} \mathbf{L} \cdot \mathbf{e}_k \tag{5.2}$$

is given in many places (e.g. Synge and Griffith, 1949). The solution in terms of Jacobi elliptic functions has three branches, depending on the values of the *kinetic energy* $E = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}$ and the magnitude of the angular momentum $L = |\mathbf{L}|$. However, if $I_2 < < I_3 < I_1$, then the component ω_3 is distinguished from the other components by having the same functional form

$$\omega_3(t) = a_3 \operatorname{sn} \tau = a_3 \operatorname{sn} \Omega(t - t_0) \tag{5.3}$$

in each branch, where a_3 and Ω are constants. The solution has been put in a 'universal form' holding for all branches by Morton, Junkins and Blanton (Table A1 of Morton *et al.*, 1974).

The problem remains to determine the attitude spinor R from the known functions $\omega_k = \omega_k(t)$. We could proceed by integrating

$$\dot{R} = \tfrac{1}{2} R i \boldsymbol{\omega} = \tfrac{1}{2} i(\omega_1 \boldsymbol{\sigma}_1 + \omega_2 \boldsymbol{\sigma}_2 + \omega_3 \boldsymbol{\sigma}_3) R,$$

but there is a much simpler way which exploits the constants of motion and determines R almost completely by algebraic means. The *angular momentum direction cosines*

$$h_k = \hat{\mathbf{L}} \cdot \mathbf{e}_k = L^{-1} I_k \omega_k \tag{5.4}$$

are more convenient parameters than the ω_k, because then we can write

$$\begin{aligned} \hat{\mathbf{L}} &= h_1 \mathbf{e}_1 + h_2 \mathbf{e}_2 + h_3 \mathbf{e}_3 = \\ &= R^\dagger (h_1 \boldsymbol{\sigma}_1 + h_2 \boldsymbol{\sigma}_2 + h_3 \boldsymbol{\sigma}_3) R = \boldsymbol{\sigma}_3, \end{aligned} \tag{5.5}$$

where we have used our prerogative to identify $\boldsymbol{\sigma}_3$ with the distinguished direction $\hat{\mathbf{L}}$ in our problem. Note that with this choice

$$h_3 = \boldsymbol{\sigma}_3 \cdot \mathbf{e}_3 = L^{-1} I_3 \omega_3 = a \operatorname{sn} \tau. \tag{5.6}$$

The question is now, what does (5.5) tell us about the functional form of R?

In the preceding section we learned that, when $\boldsymbol{\sigma}_3$ is a distinguished direction in our problem, it may be convenient to express R in terms of Cayley–Klein parameters. So, with a somewhat different notation than before, we write

$$R = \alpha_+ - i \boldsymbol{\sigma}_2 \alpha_- . \tag{5.7}$$

If we express the Cayley–Klein parameters in the 'polar form'

$$\alpha_{\pm} = \lambda_{\pm} e^{i\sigma_3 \phi_{\pm}}, \tag{5.8}$$

then we can write

$$R = \lambda_+ e^{i\sigma_3\phi_+} + i\boldsymbol{\sigma}_1 \lambda_- e^{i\sigma_3(\phi_- - (1/2)\pi)}$$
$$= e^{(1/2)i\sigma_3\phi_+}(\lambda_+ + i\boldsymbol{\sigma}_1\lambda_-)\, e^{(1/2)i\sigma_3(\phi_+ + 2\phi_- - \pi)}$$

Comparison with Equation (3.12) shows that the 'phase angles' $\phi_{\pm}$ are related to the Euler angles ψ and ϕ by

$$\psi = \phi_+, \tag{5.9}$$

$$\phi = \phi_+ + 2\phi_- - \pi. \tag{5.10}$$

Now we return to (5.5) and obtain

$$h_1\boldsymbol{\sigma}_1 + h_2\boldsymbol{\sigma}_2 + h_3\boldsymbol{\sigma}_3 = R\boldsymbol{\sigma}_3 R^{\dagger} = (\lambda_+^2 - \lambda_-^2)\boldsymbol{\sigma}_3 + 2\boldsymbol{\sigma}_1 \alpha_+ \alpha_-^{\dagger}. \tag{5.11}$$

Hence,

$$h_3 = \lambda_+^2 - \lambda_-^2 \tag{5.12}$$

and

$$h_1 + i\boldsymbol{\sigma}_3 h_2 = 2\alpha_+\alpha_-^{\dagger} = 2\lambda_+\lambda_- e^{i\sigma_3(\phi_+ - \phi_-)}. \tag{5.13}$$

Since

$$R^{\dagger}R = \lambda_+^2 + \lambda_-^2 = 1, \tag{5.14}$$

from (5.12) we obtain

$$\lambda_{\pm} = \left(\frac{1 \pm h_3}{2}\right)^{1/2}. \tag{5.15}$$

And (5.13) gives us

$$\phi_+ - \phi_- = \tan^{-1}\left(\frac{h_2}{h_1}\right), \tag{5.16}$$

Thus, we have determined $\lambda_{\pm}$ and $\phi_+ - \phi_-$ as functions of the h_k, so we can complete our solution by determining $\phi_+ + \phi_-$. That requires an integration.

Since ϕ_+ and ϕ_- are related to the Euler angles by (5.9) and (5.10), we can use (3.13) to obtain

$$\dot{\phi} + h_3\dot{\phi}_+ = \boldsymbol{\omega}\cdot\boldsymbol{\sigma}_3 = \frac{2E}{L},$$

$$h_3\dot{\phi} + \dot{\phi}_+ = \boldsymbol{\omega}\cdot\mathbf{e}_3 = \frac{Lh_3}{I_3}$$

Eliminating $\dot{\phi}_+$, we get

$$\dot{\phi} = \dot{\phi}_+ + 2\dot{\phi}_- = \frac{L}{I_3} + \left(\frac{2EI_3 - L^2}{LI_3}\right)\frac{1}{1 - h_3^2} \tag{5.17}$$

This integrates to

$$\phi(t) = \phi(t_0) + \frac{L}{I_3}(t - t_0) + \left(\frac{2EI_3 - L^2}{LI_3}\right)\left[\pi(\tau, a^2) - \pi(\tau_0, a^2)\right], \tag{5.18}$$

where

$$\pi(\tau, a^2) = \int_0^\tau \frac{d\tau}{1 - a^2 \operatorname{sn}^2 \tau} \tag{5.19}$$

is an *incomplete elliptic integral of the third kind.*

The particular parametric form for the present solution to Poinsot's problem was first found by Morton, Junkins and Blanton (Morton *et al.*, 1974). Comparison of their derivation with ours shows the felicity of geometric algebra.

6. Magnetic Resonance

Let us see how geometric algebra applies to one of the basic problems in the theory of magnetic resonance. Suppose we have an atom with intrinsic angular $\mathbf{L}$ and magnetic moment $\boldsymbol{\mu}$ related by the constitutive equation

$$\boldsymbol{\mu} = \gamma \mathbf{L}, \tag{6.1}$$

where γ is a scalar constant called the *gyromagnetic ratio*. A magnetic field $\mathbf{B}$ exerts a torque on the atom described by the dynamical equation

$$\dot{\mathbf{L}} = \boldsymbol{\mu} \times \mathbf{B} = (-\gamma\mathbf{B}) \times \mathbf{L}. \tag{6.2}$$

This implies that $\mathbf{L}^2$ is a constant of the motion, so the effect of $\mathbf{B}$ is to rotate $\mathbf{L}$. The rotation is most efficiently represented by the equation

$$\mathbf{L} = R^\dagger \mathbf{L}_0 R, \tag{6.3}$$

where $\mathbf{L}_0$ is the initial value of $\mathbf{L}$. Accordingly, we can replace (6.2) by the spinor equation of motion

$$\dot{R} = \tfrac{1}{2} R i(-\gamma\mathbf{B}) \tag{6.4}$$

subject to the initial condition $R(0) = 1$. Experimentalists wish to manipulate the magnetic moment $\boldsymbol{\mu}$ by applying suitable magnetic fields. To see how this might be done, we study the solution of (6.4) for particular applied fields.

For a static field $\mathbf{B} = \mathbf{B}_0$, the solution of (6.4) is simply

$$R = e^{-(1/2)i\gamma\mathbf{B}_0 t}, \tag{6.5}$$

This tells us that $\mathbf{L}$ and $\boldsymbol{\mu}$ precess about the magnetic field with an angular frequency $-\gamma\mathbf{B}_0$.

Now suppose we introduce a circularly polarized monochromatic plane wave propagating along the direction of the established static magnetic field $\mathbf{B}_0$. At the site of the atom, the magnetic field of such a wave is a rotating vector

$$\mathbf{b}(t) = \mathbf{b}_0 e^{i\omega t}, \tag{6.6}$$

where $\mathbf{b}_0$ and ω are constant vectors for which $\omega\mathbf{B}_0 = \mathbf{B}_0\omega$ and $\mathbf{b}_0\omega = -\omega\mathbf{b}_0$. The resultant magnetic field acting on the atom is therefore

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}_0 e^{i\omega t} = U^\dagger(\mathbf{B}_0 + \mathbf{b}_0)U, \tag{6.7}$$

where

$$U = e^{(1/2)i\omega t} \tag{6.8}$$

The form of (6.7) suggests we should write R in the factored form

$$R = SU. \tag{6.9}$$

Then, the spinor equation of motion gives us

$$\dot{R} = -\tfrac{1}{2}Ri\gamma\mathbf{B} = -\tfrac{1}{2}Si\gamma(\mathbf{B}_0 + \mathbf{b}_0)U = (\dot{S} + \tfrac{1}{2}Si\omega)U.$$

Hence S obeys the equation

$$\dot{S} = -\tfrac{1}{2}Si\gamma(\mathbf{B}_0 + \gamma^{-1}\omega + \mathbf{b}_0). \tag{6.10}$$

This has the solution

$$S = e^{-(1/2)i\gamma\mathbf{B}'t}, \tag{6.11}$$

where

$$\mathbf{B}' = \mathbf{B}_0 + \gamma^{-1}\omega + \mathbf{b}_0. \tag{6.12}$$

The motion of $\mathbf{L}$ is therefore completely described by the spinor

$$R = e^{-(1/2)i\gamma\mathbf{B}'t}\, e^{(1/2)i\omega t} \tag{6.13}$$

This tells us that the motion is a composite of two precessions with constant frequency. We may picture $\mathbf{L}$ as precessing about a static 'effective magnetic field' $\mathbf{B}'$ in a frame which itself is rotating with angular velocity $\boldsymbol{\omega}$.

Magnetic resonance is defined by the condition $\omega = -\gamma\mathbf{B}_0$. Under this condition, according to (6.11) and (6.12), $\mathbf{L}$ is precessing in the rotating frame with an angular velocity $-\gamma\mathbf{b}_0$ perpendicular to $\mathbf{B}_0$. Therefore, if $\mathbf{L}$ is initially aligned with $\mathbf{B}_0$, then its direction will be reversed in a time $T = 2\pi/\gamma\mathbf{b}_0$. Consequently, a single 'spin flip' can be produced by a pulse of duration T at resonance. See Shlichter (1963) for further discussion.

Of course, there is much more than this to the theory of magnetic resonance. But our spinor equation and its solution describe the basic idea on which the whole theory is based in a succinct and perspicuous way. Moreover, our spinor solution is closely related to the spinor wave function of an electron in quantum mechanics, though no principles peculiar to quantum mechanics have been used here.

References

Hestenes, D.: 1974, *J. Math. Phys.* **15**, 1778.
Hestenes, D.: 1983, *Celes. Mech.* **30**, 151 (this issue).
Ickes, B. P.: 1970, *AIAA J.* **8**, 13.
Morton, H. S., Junkins, J. L., and Blanton, J. N.: 1974, *Celes. Mech.* **10**, 287.
Slichter, C. P.: 1963, *Principles of Magnetic Resonance*, Harper and Row, New York, p. 18–22.
Synge, J. L. and Griffith: 1949, *Principles of Mechanics*, McGraw-Hill, New York, 2nd ed.
Whittaker, E. T.: 1944, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Dover, New York, 4th ed.