In: *Am. J. Phys.*, **47** (5), May 1979, 399–415.

# Spin and uncertainty in the interpretation of quantum mechanics

David Hestenes

A rigorous derivation of the Schrödinger theory from the Pauli (or Dirac) theory implies that the Schrödinger equation describes an electron in an eigenstate of spin. Furthermore, the ground-state kinetic energy is completely determined by the electron spin density. This can be explained by interpreting the spin as an orbital angular momentum, which is necessarily accompanied by a kinetic energy. Thus, *the spin is a zero-point angular momentum associated with the zero-point energy of the electron.* Since the dispersion in electron momentum is determined by the zero-point energy, the Heisenberg uncertainty relations for an electron can be interpreted as a property of the electron spin motion. The kinetic interpretation of spin and the statistical interpretation of quantum mechanics can be jointly sustained by regarding the electron as a point particle. It follows that stationary electron states are sources of fluctuating electric fields. There is reason to believe that these fluctuating fields are responsible for the Van der Waals force and can be identified with electromagnetic vacuum field fluctuations. The kinetic interpretation of spin then implies that the Van der Waals forces are spin dependent. These ideas are not only consistent with the conventional mathematical formalism of quantum mechanics, they provide a more complete and coherent interpretation of many details in the formalism than does the alternative Copenhagen interpretation. They present some difficulties, however, which, if the kinetic interpretation of spin is correct, probably require some modification of quantum electrodynamics to be resolved.

## 1. INTRODUCTION

The longstanding debate and discussion about the interpretation of quantum mechanics has been strongly focused on the meaning of Heisenberg's *Uncertainty Relations* for the position and momentum of an *electron*. However, the various interpretations which have been proposed and widely accepted all suffer from a serious drawback: they are limited to the Schrödinger theory and fail to take into account either spin or relativity. We shall see, in fact, that spin cannot be ignored.

To ascertain the role of spin, we will examine the Uncertainty Relations within the context of the Pauli electron theory.[1,2] To be sure, the analysis should be extended to the Dirac theory, but it is best to avoid the complications of relativity and many particle effects at this preliminary stage. However, to insure that our conclusions are not inconsistent with the Dirac theory, we use an interpretation of the Pauli theory which is derived from a standard interpretation of the Dirac theory.[3] This point is crucial, because it provides a solid foundation for some judgments which are widely at variance with more or less standard interpretations of nonrelativistic quantum mechanics.

Just as the interpretation of the Pauli theory is determined by its relation to the Dirac theory, so the interpretation of the Schrödinger theory is determined by its relation to the Pauli theory, at least when the theory is applied to an electron rather than a spinless particle like a pi meson. This elementary point of logical consistency has been overlooked or ignored in every discussion of the Uncertainty Relations and the interpretation of nonrelativistic quantum mechanics. It is a nontrivial point, for, as we shall see, it radically alters the conventional interpretation of the Schrödinger theory.

In the absence of a magnetic field, the Pauli equation reduces to the Schrödinger equation. The Pauli wave function can then be expressed as a superposition of two orthogonal spin eigenstates which are not coupled, so the Schrödinger equation describes the time evolution of a single spin eigenstate. Thus, to be consistent with the Pauli theory, the Schrödinger theory must be regarded as describing an electron in a eigenstate of spin and not, as universally supposed, an electron without spin. The solutions of the Schrödinger equation are, of course, not affected by this point, but the interpretation of the solutions is greatly affected. To emphasize the fact that the Schrödinger electron theory must be regarded as identical to the Pauli electron theory in the absence of magnetic fields, we refer to the theory as the *Pauli-Schrödinger* (PS) *electron theory* and to the wave equation as the PS equation.

The interpretation of the Pauli theory depends critically on the role of spin. What, precisely, does it mean to say that the electron has an *intrinsic spin?* The mathematics of spin is well defined and well understood, but an interpretation seeks to go beyond the mathematics and establish a coherent model of objective reality. Two kinds of model for the electron have been discussed in the literature. The most common kind of model pictures the electron as an extended (usually rigid) body with an internal angular momentum to be identified with the spin. All such models are inconsistent with the mathematical structure of the Pauli theory, so we will not bother to analyze them here. Because these models fail, they are usually introduced only for heuristic purposes, and it is said that "spin is a quantum-mechanical phenomenon without classical analog."

Instead of regarding spin as a property of electron structure, we shall examine the possibility of interpreting spin as a dynamical property of electron motion. An interpretation of this kind has been suggested from time to time in connection with the Dirac theory,[30,31] especially as an interpretation of the so-called "zitterbewegnung." However, its implications for the Pauli theory have not been examined. We shall see that it provides a more complete and coherent interpretation of mathematical details in the PS theory than can be achieved by the alternatives such as the Copenhagen interpretation.

To say that the electron spin and magnetic moment are dynamical properties is to hold that they describe a circulation of electron mass and charge. To justify and delineate this interpretation, we must examine the flow of electron charge and mass in the PS theory

and its relation to the spin density; then we must show that the theory associates a kinetic energy with this flow, and that all expection values have a sensible interpretation. This task is carried out in Sec. II, though, to keep questions of interpretation to the fore, mathematical details have been relegated to an appendix.

Section III analyzes wave packet solutions of the PS equation for a free particle and a harmonic oscillator. The packets are found to be *spinning Gaussian balls*, with rotational kinetic energy and angular momentum related to Heisenberg's uncertainty relations in an essential way.

A fully satisfactory statistical interpretation of quantum mechanics would require that terms describing energy fluctuations in the Pauli theory be unambiguously identified. Section IV discusses how this might be accomplished by a phase-space formulation of quantum mechanics. It is pointed out that Wigner's phase-space formulation is incompatible with the notion that spin has a dynamical origin. It is suggested that eigenfunctions in the PS theory can be identified with statistical ensembles for which the corresponding eigenvalue is a common property of each member. Finally, the relation between zero-point angular momentum and zero-point energy is established by elementary arguments, which can be compared with similar arguments based on the uncertainty relations.

Section V discusses the incompatibility of the statistical interpretation of quantum mechanics with the notion of wave-particle duality. In defense of the view that the electron can always be regarded as a particle, the explanation of diffraction by quantized momentum exchange is reviewed. It is pointed out that the kinetic interpretation of spin implies that there is an intrinsic periodicity of electron motion even though the electron is assumed to be a particle.

Section VI points out that the statistical interpretation implies that stationary states are sources of fluctuating electric fields. The possibility of identifying these fluctuating fields with Van der Waals forces and electromagnetic vacuum fluctuations is explored. The problem of explaining why stationary states are radiationless is raised, and some speculations as to how it might be resolved are offered. It is suggested that the electron spin like the electron mass arises from the electromagnetic self-interaction and that the Pauli principle might be explained as a condition for radiationless motion.

Section VII suggests two ways that the algebraic structure of the PS theory, as reflected in the role of complex numbers, might fit into the general algebraic structure of elementary particle theory.

A summary of the main argument is given in Sec. VIII.

## II. CURRENTS AND EXPECTATION VALUES

The physical interpretation of the PS theory depends critically on correspondences of the wave function with observables, above all, on the identification of *currents* describing the flow of charge and momentum. There are several possibilities. Our aim is to determine that identification which provides the simplest and most comprehensive interpretation of the entire PS theory in accord with physical facts.

In the conventional formulation of the Schrödinger electron theory, the probability density $\rho$ and the probability current $\rho\mathbf{u}$ are determined by the wave function according to the rules

$$\rho = \Psi^\dagger \Psi \,, \tag{2.1a}$$

$$\rho u_k = -(i'\hbar/2m)(\Psi^\dagger \partial_k \Psi - \partial_k \Psi^\dagger \Psi) - (e/mc)A_k \Psi^\dagger \Psi \,. \tag{2.1b}$$

Exactly the same identifications are commonly made in the Pauli electron theory, with $\Psi$ being the two component Pauli wave function.[1] This is as it should be, if the Schrödinger theory is to be regarded as a special case of the Pauli theory.

In the Schrödinger theory the charge current is identified as $e\rho\mathbf{u}$, with little justification beyond the fact that this is consistent with charge conservation. In the Pauli theory $e\rho$ is also identified as charge density, but the *Schrödinger current* $e\rho\mathbf{u}$ is supplemented by a *magnetization current* $c\boldsymbol{\nabla} \times \mathbf{m}$ to get the total charge current

$$\mathbf{j} = e\rho\mathbf{u} + c\boldsymbol{\nabla} \times \mathbf{m} \,, \tag{2.2a}$$

where the magnetic moment density is proportional to a spin density $\rho\mathbf{s}$ determined by the wave function according to the rule

$$m_k = (e\hbar/2mc)\Psi^\dagger \sigma_k \Psi = (e/mc)\rho s_k \,. \tag{2.2b}$$

Equation (2.2a) is commonly interpreted as a partition of the total current into a convective current $e\rho\mathbf{u}$ due to a flow of charge and a magnetization current arising from an "intrinsic magnetic moment," presumably a structural property of the electron.[1] We shall examine an alternative interpretation.

Defining the *Pauli current* $\rho\mathbf{v}$ by writing $\mathbf{j} = e\rho\mathbf{v}$, Eq. (2.2a) can be put in the form

$$m\rho\mathbf{v} = m\rho\mathbf{u} + \boldsymbol{\nabla} \times (\rho\mathbf{s}) \,, \tag{2.3}$$

suggesting that either $m\rho\mathbf{v}$ or $m\rho\mathbf{u}$ should be interpreted as the momentum density of the electron. We shall see that there are good reasons to believe that the Pauli current rather than the Schrödinger current correctly describes both the momentum density and the convective flow of electron charge. Before considering evidence within the PS theory itself, it is worth knowing that this viewpoint is fully in accord with the conventional interpretation of the Dirac electron theory. Equation (2.3) is the nonrelativistic limit of an equation in the Dirac theory, the *Pauli current* being the nonrelativistic limit of the usual *Dirac current*, while the *Schrödinger current* is the nonrelativistic limit of the so-called *Gordon current* (see Ref. 3). The Dirac current plays a central role in the Dirac theory, while the Gordon current is only a minor subsidiary. This is reason enough to regard the Pauli current as more fundamental than the Schrödinger current, but let us see the basis for this view within the Pauli theory.

The question is, on what evidence can $\mathbf{v}$ rather than $\mathbf{u}$ be identified as the velocity of the electron? Unfortunately, we cannot answer this question directly by measuring the local flow of electron charge and momentum in an atom. We are limited to indirect inferences from measurements of average values. We cannot distinguish between $\mathbf{v}$ and $\mathbf{u}$ by measuring the average electron velocity, for it follows from Eq. (2.3) that:

$$\langle\, \mathbf{v} \,\rangle = \langle\, \mathbf{u} \,\rangle \,. \tag{2.4}$$

However, some distinction between $\mathbf{v}$ and $\mathbf{u}$ can be made by measuring higher-order moments with respect to the probability distribution. In particular, considering Eq. (A31) in the appendix, we see that Eq. (2.3) implies that

$$\langle\, \mathbf{r} \times m\mathbf{v} \,\rangle = \langle\, \mathbf{r} \times m\mathbf{u} \,\rangle + 2\langle\, \mathbf{s} \,\rangle \,. \tag{2.5}$$

It can be seen from Eq. (A30) that $\langle\, \mathbf{r} \times m\mathbf{u} \,\rangle$ is the expected angular momentum as defined by the usual "orbital angular momentum operators" in the Schrödinger and Pauli theories. We shall deemphasize the conventional "operator language" in quantum mechanics, because it obscures the relation of expectation values to the Pauli and Schrödinger currents which we wish to consider seriously as candidates for an actual local average flow of charge and mass. Indeed, Eq. (2.5) reveals that the conventional angular momentum operators have not been correctly identified if $\mathbf{v}$ is the "actual" electron velocity as suggested by the Dirac theory.

As shown in every text in quantum mechanics, the probability distribution for the ground state of a hydrogenlike atom is spherically symmetric, and it is assumed to be static, because the Schrödinger current vanishes. However, Eq. (2.3) implies that the Pauli current does not vanish in this case. The Pauli current describes a circulation of the ground state density distribution with an average angular momentum $\langle\, \mathbf{r} \times m\mathbf{v} \,\rangle = 2\langle\, \mathbf{s} \,\rangle$. This result is to be regarded as a feature of the Schrödinger electron theory, if the Schrödinger theory describes an electron in an eigenstate of spin, as we have already argued it must. Appendix A explains that an electron is in a spin eigenstate if and only if the spin vector $\mathbf{s}$ is constant. But $|\, \mathbf{s} \,| = \frac{1}{2}\hbar$, so the Pauli current assigns to the hydrogen ground state an average angular momentum with magnitude $|\, \langle\, \mathbf{r} \times m\mathbf{v} \,\rangle \,| = |\, 2\mathbf{s} \,| = \hbar$. Is it a mere coincidence that this assignment agrees exactly with the one made in the Bohr theory and that, as the textbooks show, the ground state radial distribution function in the Schrödinger theory has its maximum at the Bohr radius?

We see from this example, that if $\rho m\mathbf{v}$ is identified as the electron momentum density, the interpretation of quantum theory must be quite different from the conventional one. On this interpretation, the electron angular momentum is entirely orbital. The electron spin is to be interpreted not as a structural property of the electron but as a dynamical property of electron motion. If the electron is to be regarded as having an intrinsic angular momentum, it would seem better to identify this quantity as twice the spin $2\mathbf{s}$ rather than the spin itself, because this is the ground state angular momentum of the electron. The electron ground-state charge distribution in hydrogen is "spinning," but this hardly suggests a "spinning ball" model for the electron, for the nucleus is at the center of the spinning distribution. It is simpler to interpret the Pauli current $\rho\mathbf{v}$ as the local average velocity of a point particle circulating about the nucleus. The particle interpretation implies a circulating current with constant charge to mass ratio, that is, unless there is a hidden source of momentum, the local charge current must differ from the momentum density only by the factor $e/m$.

To see if it is reasonable to interpret the Pauli current as both mass and charge currents, we must examine the interaction with a magnetic field. From the Pauli equation (A36), we find via (A38) that the average energy can be expressed as a sum of kinetic and potential energies;

$$\langle\, E \,\rangle = (2m)^{-1}\langle\, \hat{\mathbf{p}}^2 \,\rangle + 2\langle\, e\varphi \,\rangle\,. \tag{2.6}$$

According to (A33), the expectation value of the Pauli kinetic-energy operator $(2m)^{-1}\langle\, \hat{\mathbf{p}}^2 \,\rangle$ differs from that of the Schrödinger kinetic-energy operator $(2m)^{-1}\langle\, \hat{p}^2 \,\rangle$ by

$$(1/2m)\langle\, \hat{\mathbf{p}}^2 \,\rangle = (1/2m)\langle\, \hat{p}^2 \,\rangle - (e/mc)\langle\, \mathbf{B}\cdot\mathbf{s} \,\rangle\,. \tag{2.7}$$

The last term here has long been regarded as the potential energy of interaction of an intrinsic electron magnetic moment with the magnetic field. If this be correct, then it

is a mistake to interpret $(2m)^{-1}\hat{\mathbf{p}}^2$ as a kinetic energy operator. However, it should be noted that the conventional interpretation of (2.7) tacitly assumes that the kinetic energy operator for a spinless particle has been correctly identified as $(2m)^{-1}\hat{p}^2$, whereas we have seen that it is logically inconsistent to interpret the Schrödinger theory as a description of a spinless electron even in the absence of a magnetic field.

Any expression identified as the electron kinetic energy must be consistent with the identification of the electron momentum. To see how the kinetic-energy term (2.7) relates to our candidate for the local electron velocity $\mathbf{v}$ given by the Pauli current $\rho\mathbf{v}$, we eliminate the operators from (2.7), re-expressing it entirely in terms of observables. From Eq. (A34) obtain

$$\frac{1}{2m}\langle\,\hat{\mathbf{p}}^2\,\rangle = \frac{1}{2m}\langle\,\hat{\mathbf{P}}^2\,\rangle - \frac{e}{c}\langle\,\mathbf{A}\cdot\mathbf{v}\,\rangle - \frac{e^2}{2mc^2}\langle\,\mathbf{A}^2\,\rangle\,, \tag{2.8}$$

where $(2m)^{-1}\hat{\mathbf{P}}^2 = (-\hbar^2/2m)\boldsymbol{\nabla}^2$ is the kinetic-energy operator in the absence of any magnetic interaction. In the first approximation, the wave function is not altered by a perturbing magnetic field and the complete change in electron energy due to the field is given by the linear interaction term in (2.8). For a constant magnetic field $\mathbf{B}$ with vector potential $\mathbf{A} = (1/2)\mathbf{r}\times\mathbf{B}$, the term becomes

$$(-e/c)\langle\,\mathbf{A}\cdot\mathbf{v}\,\rangle = (e/2mc)\mathbf{B}\cdot\langle\,\mathbf{r}\times m\mathbf{v}\,\rangle\,. \tag{2.9}$$

This will be recognized as identical to the classical expression for the precession energy of an orbiting point charge induced by a magnetic field (Larmor's theorem). It describes a change in the electron's kinetic energy rather than its potential energy as attributed to the spin interaction term in (2.7). Of course the "spin term" in (2.7) has been incorporated into the "orbital term" (2.9) by "properly" identifying the electron velocity. We conclude that our interpretation of the Pauli current is completely consistent with the identification of the Pauli kinetic-energy operator.

To eliminate operators completely from the kinetic energy term we use Eq. (A35), with the result

$$(1/2m)\langle\,\hat{\mathbf{p}}^2\,\rangle = (1/2)m\langle\,\mathbf{v}^2 + c^2\beta^2\,\rangle\,, \tag{2.10}$$

where, according to (A28),

$$mc\rho\beta = -\boldsymbol{\nabla}\cdot(\rho\mathbf{s})\,. \tag{2.11}$$

In the expression for kinetic energy on the right side of (2.10), the magnetic field does not appear explicitly at all. This is as it should be, because the magnetic field affects the energy of a particle only by changing the value of its velocity. The magnetic field appears explicitly in the energy only when the *change* in velocity it induces is evaluated, as in Eqs. (2.8) and (2.9).

The unsolicited dimensionless function $\beta$ in (2.10) provides the major obstacle to a complete interpretation of the Pauli theory. What is the significance of its contribution to the kinetic energy? We can gain some insight into this question by again considering an electron in the ground state of atomic hydrogen. In this case, $\mathbf{v} = (m\rho)^{-1}\boldsymbol{\nabla}\times\rho\mathbf{s}$, and (2.10) can be written

$$(1/2m)\langle\,\hat{\mathbf{p}}^2\,\rangle = (1/2m)\langle\,(\rho^{-1}\boldsymbol{\nabla}\times\rho\mathbf{s})^2 + (\rho^{-1}\boldsymbol{\nabla}\cdot\rho\mathbf{s})^2\,\rangle\,. \tag{2.12}$$

This shows that the entire ground-state kinetic energy arises from the spin density. Part of this energy is kinetic energy of the circulating electron; this energy of the "magnetization current" is manifested in the electron's magnetic moment, which is accessible to fairly direct experimental manipulation by magnetic resonance techniques. The physical significance of the kinetic energy determined by $\beta$ is not so evident. But the symmetry of contributions from $\boldsymbol{\nabla}\times\rho\mathbf{s}$ and $\boldsymbol{\nabla}\cdot\rho\mathbf{s}$ to (2.12) suggests some physical connection between them.

When the electron is in an eigenstate of spin, the spin vector $\mathbf{s}$ is constant and (2.12) can be put in the form

$$\begin{aligned}\langle\,\hat{\mathbf{p}}^2\,\rangle &= \langle\,(\rho^{-1}\mathbf{s}\times\boldsymbol{\nabla}\rho)^2+(\rho^{-1}\mathbf{s}\cdot\boldsymbol{\nabla}\rho)^2\,\rangle \\ &= \langle\,\mathbf{s}^2(\rho^{-1}\boldsymbol{\nabla}\rho)^2\,\rangle = (\hbar^2/4)\langle\,\boldsymbol{\nabla}\ln\rho)^2\,\rangle\,. \end{aligned} \tag{2.13}$$

This result also applies when $\mathbf{s}$ is precessing in a constant magnetic field. The dependence of the ground-state energy on spin is well disguised on the right side of (2.13), so well disguised, in fact, that no one suspected a connection with spin when such an expression appeared in the Schrödinger theory.

Equation (2.13) can be regarded as version of Heisenberg's uncertainty relation. From Schrödinger's groundstate hydrogen wave function we find $(\boldsymbol{\nabla}\ln\rho)^2=4/a^2$ and $\langle\,\mathbf{r}^2\,\rangle=3a^2$, where $a$ is the Bohr radius. We have $\langle\,\hat{\mathbf{p}}\,\rangle=0$ and $\langle\,\mathbf{r}\,\rangle=0$ as well, so from (2.13) we get the "uncertainty relation"

$$\langle\,\hat{\mathbf{p}}^2\,\rangle\langle\,\mathbf{r}^2\,\rangle = (\langle\,\hat{\mathbf{p}}^2\,\rangle-\langle\,\hat{\mathbf{p}}\,\rangle^2)(\langle\,\mathbf{r}^2\,\rangle-\langle\,\mathbf{r}\,\rangle^2) = 3\hbar^2\,. \tag{2.14}$$

From the spherical symmetry of $\rho$, it is obvious that

$$\begin{aligned}\langle\,(\rho^{-1}\mathbf{s}\cdot\boldsymbol{\nabla}\rho)^2\,\rangle &= (1/3)\mathbf{s}^2\langle\,(\boldsymbol{\nabla}\ln\rho)^2\,\rangle \\ &= (1/2)\langle\,\rho^{-1}\mathbf{s}\times\boldsymbol{\nabla}\rho)^2\,\rangle\,. \end{aligned} \tag{2.15}$$

Therefore, exactly one-third of the ground-state energy and one-third of the "uncertainty" is due to the spin divergence function $\beta$.

Now let us consider the so-called "Ehrenfest theorem" in the Pauli theory, to see how critically its interpretation depends on the identification of observables. The object of this theorem is to derive an equation of motion for the average electron velocity and interpret this as an equation for a classical particle. If we suppose that the Schrödinger current $\rho\mathbf{u}$ describes the electron velocity, then from Eq. (A39b) we get

$$\frac{d}{dt}\langle\,m\mathbf{u}\,\rangle = \Big\langle\,e\mathbf{E}+\frac{e}{c}\mathbf{u}\times\mathbf{B}+\frac{e}{mc}\boldsymbol{\nabla}'\mathbf{B}'\cdot\mathbf{s}\,\Big\rangle\,, \tag{2.16}$$

where the primes in $\boldsymbol{\nabla}'\mathbf{B}'\cdot\mathbf{s}$ are meant to indicate that the gradient operator $\boldsymbol{\nabla}$ differentiates $\mathbf{B}$ but not $\mathbf{s}$. The last term in (2.16) appears to be just the "Stern-Gerlach force" on a particle with intrinsic magnetic moment $\mu=(e/mc)\mathbf{s}$. On the other hand, if we suppose that the Pauli current $\rho\mathbf{v}$ describes the electron velocity, then from Eq. (A41b) we get

$$\frac{d}{dt}\langle\,m\mathbf{v}\,\rangle = \Big\langle\,e\mathbf{E}+\frac{e}{c}\mathbf{v}\times\mathbf{B}\,\Big\rangle\,, \tag{2.17}$$

in which no intrinsic magnetic moment is evident. Of course Eqs. (2.16) and (2.17) are completely equivalent; the left sides are equivalent by (2.4), while a direct proof that the right sides are equivalent is obtained by using (A43) to establish the identity

$$\langle\, \mathbf{u}\times\mathbf{B} + m^{-1}\boldsymbol{\nabla}'\mathbf{B}'\cdot\mathbf{s}\,\rangle = \langle\, \mathbf{v}\times\mathbf{B}\,\rangle\,. \tag{2.18}$$

Evidently, an effective magnetic moment arises from (2.17) by a circulation of the Pauli current about the center of charge (or mass) moving with velocity $\langle\,\mathbf{v}\,\rangle$. The spin density $\rho\mathbf{s}$ determines this circulation locally, so we can regard it as describing a property of electron motion rather than electron structure.

From Eq. (A39c) or (A41c), we find that the average spin vector obeys the equation of motion

$$\frac{d}{dt}\langle\,\mathbf{s}\,\rangle = \frac{e}{mc}\langle\,\mathbf{s}\times\mathbf{B}\,\rangle\,. \tag{2.19}$$

This is the fundamental equation of electron spin resonance theory. Since the erroneous view that the electron spin can only be found pointing either parallel or antiparallel to the quantizing magnetic field is still occasionally found in the literature, it may be worth mentioning that Eq. (2.19) completely disproves this view.[5] However, a point of interest here is that for the ground state of a hydrogenlike atom in a uniform magnetic field, Eq. (2.5) enables us to reexpress (2.19) as an equation for the precession of the average orbital angular momentum;

$$\frac{d}{dt}\langle\,\mathbf{r}\times m\mathbf{v}\,\rangle = \frac{e}{mc}\langle\,\mathbf{r}\times m\mathbf{v}\,\rangle\times\mathbf{B}\,. \tag{2.20}$$

The curious fact here is that the precession frequency is the same as the classical "cyclotron frequency" $(-e/mc)\mathbf{B}$ of a "free particle" instead of the Larmor frequency $(-e/2mc)\mathbf{B}$ expected for a bound particle.

Our analysis of the PS theory to this point suggests the need for a fairly extensive reinterpretation of results from Schrödinger theory, especially in the assignment of angular momentum quantum numbers in spectroscopy. However, it has not been suggested that the validity of those results be questioned, so such a reinterpretation holds little interest, except where it bears on critical issues such as the nature of the ground-state energy and angular momentum.

## III. SPINNING WAVE PACKETS

Let us apply what we have learned to interpret the motion of a wave packet. We can decompose the motion of the packet into a motion of its *centroid* $\langle\,\mathbf{x}\,\rangle$ and an "internal motion" relative to the centroid. The centroid can be found at any time from a given initial value and average velocity $\langle\,\mathbf{v}\,\rangle$ by integrating

$$\langle\,\mathbf{v}\,\rangle = \langle\,\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{x}\,\rangle = \frac{\langle\, d\mathbf{x}\,\rangle}{dt} = \frac{d\langle\,\mathbf{x}\,\rangle}{dt}\,. \tag{3.1}$$

An interpretation of the internal motion comes from supposing that the Pauli current $\rho\mathbf{v}$ gives us the local electron velocity $\mathbf{v}$.

As explained in the Appendix, an electron in an eigenstate of spin has a constant local spin vector

$$\mathbf{s} = (1/2)\hbar\boldsymbol{\sigma}_3\,, \tag{3.2}$$

and the solution of the PS equation can be put in the form

$$\Psi = \rho^{1/2} e^{i\boldsymbol{\sigma}_3\theta}\,. \tag{3.3}$$

This differs from a conventional solution of Schrödinger's equation only in the use of a unit imaginary $i\boldsymbol{\sigma}_3$ with a geometrical interpretation to describe the spin direction.

A solution of the PS equation is said to be a Gaussian wave packet if the probability density has the form

$$\rho = \Psi^\dagger\Psi = (2\pi a^2)^{-3/2} e^{-\mathbf{r}^2/2a^2}\,, \tag{3.4}$$

where $\mathbf{r} \equiv \mathbf{x} - \langle\,\mathbf{x}\,\rangle$, so

$$\langle\,\mathbf{r}^2\,\rangle = \langle\,(\mathbf{x} - \langle\,\mathbf{x}\,\rangle)^2\,\rangle = 3a^2\,. \tag{3.5}$$

Free particle solutions of the Schrödinger equation are usually built up as a superposition of plane waves, but Synge[6] has shown how to get the general Gaussian solution much more simply and directly. We merely record his result and use it. His solution can be put in the form (3.3), where

$$\theta = \frac{3}{2}\tan^{-1}\left(\frac{\tau}{t}\right) + \frac{1}{2}\frac{m}{\hbar}\left(\frac{t\mathbf{r}^2}{t^2+\tau^2} + 2\mathbf{r}\cdot\langle\,\mathbf{v}\,\rangle + t^2\langle\,\mathbf{v}\,\rangle^2\right) + \theta_0\,, \tag{3.6}$$

and $\rho$ is given by (3.4) with

$$a^2 = (\hbar/2m\tau)(t^2+\tau^2)\,. \tag{3.7}$$

The scalars $\theta_0$ and $\tau$ in (3.7) are disposable constants. Equation (2.17) tells us that $\langle\,\mathbf{v}\,\rangle$ is constant, and Eq. (3.1) gives

$$\langle\,\mathbf{x}\,\rangle = \langle\,\mathbf{v}\,\rangle t\,. \tag{3.8}$$

Other properties of the packet can now be found by direct calculation.

First we consider the free particle currents. From the definition (A20) of the Schrödinger current $\rho\mathbf{u}$, it follows that, for a free particle:

$$\mathbf{u} = (\hbar/m)\boldsymbol{\nabla}\theta = \langle\,\mathbf{v}\,\rangle + [\mathbf{r}t/(t^2+\tau^2)]\,. \tag{3.9}$$

From (3.4) we find

$$\boldsymbol{\nabla}\ln\rho = -a^{-2}\mathbf{r}\,, \tag{3.10}$$

so

$$\rho^{-1}\boldsymbol{\nabla}\times(\rho\mathbf{s}) = -\mathbf{s}\times\boldsymbol{\nabla}\ln\rho = a^{-2}\mathbf{s}\times\mathbf{r}\,. \tag{3.11}$$

Inserting these results in (2.3), we find that the *local electron velocity* is given by

$$\mathbf{v} = \mathbf{u} + \boldsymbol{\omega}\times\mathbf{r} = \langle\,\mathbf{v}\,\rangle + [\mathbf{r}t/(t^2+\tau^2)] + \boldsymbol{\omega}\times\mathbf{r}\,, \tag{3.12a}$$

where

$$\boldsymbol{\omega} = \mathbf{s}/ma^2 = \tau\boldsymbol{\sigma}_3/(t^2+\tau^2)\,. \tag{3.12b}$$

From Eq. (3.9) we find

$$\langle\, \mathbf{x} \times m\mathbf{u} \,\rangle = \langle\, \mathbf{x} \,\rangle \times \langle\, m\mathbf{v} \,\rangle\,, \tag{3.13}$$

and from (2.5), or by direct calculation from (3.12), we find that the average "internal angular momentum" has the constant value

$$\langle\, \mathbf{r} \times m\mathbf{v} \,\rangle = \langle\, \mathbf{x} \times m\mathbf{v} \,\rangle - \langle\, \mathbf{x} \,\rangle \times \langle\, m\mathbf{v} \,\rangle = 2\mathbf{s} = \hbar\boldsymbol{\sigma}_3\,. \tag{3.14}$$

Now we are in position to describe the kinematics of the wave packet in detail.

According to (3.4), (3.5), (3.7), and (3.8), the density of the wave packet is a "Gaussian ball" expanding with time about its centroid $\langle\, \mathbf{x} \,\rangle$ which moves with constant velocity $\langle\, \mathbf{v} \,\rangle$. According to (3.12a), the expansion proceeds uniformly at all points of the packet with radial velocity $\mathbf{r}t(t^2+\tau^2)^{-1}$. This much conforms to the conventional description of a free particle wave packet. However, Eq. (3.12) also tells us that the wave packet is spinning with angular velocity $\boldsymbol{\omega}$ about a fixed axis through the centroid. The "moment of inertia" of the spinning wave packet increases as the packet expands, but, according to (3.12b) and (3.14), this is compensated by a decrease of the angular velocity which keeps the angular momentum of the packet constant.

The "streamlines" of local particle flow can be "read off" directly from (3.12). In the plane $\mathbf{r}\cdot\boldsymbol{\omega} = 0$ through the centroid, the streamlines are expanding spirals. In general, the streamlines circulate on cones centered at the centroid, with the limiting case of pure expansion along the axis of rotation.

It will be noted in (3.12a) that the relative velocity $\mathbf{v} - \langle\, \mathbf{v} \,\rangle$ exceeds the velocity of light for sufficiently large $r = |\,\mathbf{r}\,|$. This is allowable in a nonrelativistic theory, and it can be shown that the difficulty is resolved in the relativistic Dirac theory. Nevertheless, it is worth noting that the Pauli current has a unique maximum. According to (3.4) and (3.12a), we have $|\,\rho(\mathbf{v} - \langle\, \mathbf{v} \,\rangle)\,| \sim |\,\mathbf{r}\,|\exp(-\mathbf{r}^2/2a^2)$, and this has its maximum value at $\mathbf{r}^2 = a^2$. From (3.12a) and (3.7) we see that the relative velocity at this radius has its maximum magnitude in the plane $\mathbf{r}\cdot\boldsymbol{\omega} = 0$ , with a value given by $(\mathbf{v} - \langle\, \mathbf{v} \,\rangle)^2 = \hbar/2m\tau$. The curious fact is that this value is independent of time, and we will shortly see the same number appear in a slightly different context.

Now let us examine the contribution of internal motion in the packet to the total energy of the electron. According to (2.10), the total energy is given by

$$\langle\, E \,\rangle = (1/2m)\langle\, \hat{\mathbf{p}}^2 \,\rangle = (1/2)m\langle\, \mathbf{v}^2 + c^2\beta^2 \,\rangle\,. \tag{3.15}$$

From (3.12a) we find that kinetic energy is composed of the independent contributions

$$\langle\, \mathbf{v}^2 \,\rangle = \langle\, \mathbf{v} \,\rangle^2 + \langle\, [t\mathbf{r}/(t^2+\tau^2)]^2 \,\rangle + \langle\, (\boldsymbol{\omega} \times \mathbf{r})^2 \,\rangle\,. \tag{3.16}$$

Using (3.5) and (3.7) to evaluate the kinetic energy of expansion, we get

$$\frac{m}{2}\left\langle \left(\frac{t\mathbf{r}}{t^2+\tau^2}\right)^2 \right\rangle = \frac{3}{4}\frac{\hbar}{\tau}\left(\frac{t^2}{t^2+\tau^2}\right)\,. \tag{3.17a}$$

The rotational kinetic energy of the wave packet is

$$\begin{aligned}\frac{m}{2}\langle\, (\boldsymbol{\omega} \times \mathbf{r})^2 \,\rangle &= \frac{1}{2m}\langle\, (\rho^{-1}\boldsymbol{\nabla} \times \rho\mathbf{s})^2 \,\rangle \\ &= \frac{1}{2ma^4}\langle\, (\mathbf{s} \times \mathbf{r})^2 \,\rangle = \frac{\hbar}{2\tau}\left(\frac{\tau^2}{t^2+\tau^2}\right)\,.\end{aligned} \tag{3.17b}$$

The remaining contribution to the energy is

$$\frac{m}{2}\langle\, c^2\beta^2\,\rangle = \frac{1}{2m}\langle\, (\rho^{-1}\boldsymbol{\nabla}\cdot\rho\mathbf{s})^2\,\rangle$$
$$= \frac{\langle\, (\mathbf{s}\cdot\mathbf{r})^2\,\rangle}{2ma^4} = \frac{\hbar}{4\tau}\left(\frac{\tau^2}{t^2+\tau^2}\right). \tag{3.17c}$$

Adding all contributions, we find that the total energy has the value

$$\langle\, E\,\rangle = \frac{1}{2m}\langle\, \hat{\mathbf{p}}^2\,\rangle = \frac{1}{2}m\langle\, \mathbf{v}^2\,\rangle + \frac{3}{4}\frac{\hbar}{\tau}. \tag{3.18}$$

The last term in (3.18) is the total internal energy of the packet, but most interesting is the decomposition of this energy into the parts (3.17a)–(3.17c). The expansion energy (3.17a) can be decomposed into two independent parts, an energy of "axial expansion" along the spin axis

$$\frac{m}{2}\left\langle \left(\frac{t\mathbf{r}\cdot\boldsymbol{\sigma}_3}{t^2+\tau^2}\right)^2\right\rangle = \frac{\hbar}{4\tau}\left(\frac{t^2}{t^2+\tau^2}\right), \tag{3.19a}$$

and an energy of "radial expansion" perpendicular to the spin axis

$$\frac{m}{2}\left\langle \left(\frac{t\mathbf{r}\times\boldsymbol{\sigma}_3}{t^2+\tau^2}\right)^2\right\rangle = \frac{\hbar}{2\tau}\left(\frac{t^2}{t^2+\tau^2}\right), \tag{3.19b}$$

The radial expansion energy (3.19b) adds to the rotational kinetic energy (3.19b) to produce a constant internal energy $\hbar/2\tau$. At time $t = 0$ this energy is all in the rotational mode, but as it increased the energy is gradually shifted to the expansion mode. The wave packet with minimum radius (at $t = 0$) is therefore characterized by having all of this internal energy in the rotational mode.

In a similar way the axial expansion energy (3.19a) combines with the "spin divergence" energy (3.17c) producing a constant internal energy $\hbar/4\tau$. However, this quantity does not have an obvious interpretation as energy of electron motion. Since it is independent of the rotational mode, perhaps it could be eliminated without affecting predictions of the theory. In contrast to the rotational mode, it does not seem to be related to observable properties of the electron magnetic moment. Its only apparent function here is to retain the spherical symmetry of the probability distribution.

Our results can be used to derive and interpret the Heisenberg uncertainty relations for a free electron. Since $\langle\, m\mathbf{v}\,\rangle = \langle\, \hat{\mathbf{p}}\,\rangle$ by (A48), we can rewrite (2.17) in the form

$$3(\Delta p_x)^2 \equiv \langle\, \hat{\mathbf{p}}^2\,\rangle - \langle\, \hat{\mathbf{p}}\,\rangle^2 = 3m\hbar/2\tau\,, \tag{3.20}$$

where $\Delta p_x$ is supposed to be the "dispersion in the $x$ component of momentum." From (3.5) and (3.7) we get

$$3(\Delta x)^2 \equiv \langle\, \mathbf{x}^2\,\rangle - \langle\, \mathbf{x}\,\rangle^2 = 3a^2 = (3\hbar\tau/2m)(1 + t^2/\tau^2)\,, \tag{3.21}$$

where $\Delta x$ is supposed to be the "dispersion in the $x$ coordinate" of the electron. Multiplying (3.20) and (3.21) we get the conventional form for the uncertainty relation

$$\Delta x\Delta p_x = (\hbar/2)(1 + t^2/\tau^2)^{1/2}\,. \tag{3.22}$$

We have seen that the dispersion of momentum can be attributed primarily if not entirely to internal motion of the wave packet, so the uncertainty relations can also be attributed to this internal motion.

The uncertainty relation (3.22) is supposed to be *isotropic* in the sense that the rectangular coordinate $x$ can refer to any direction in space. However, the interpretation of the uncertainty relation is seen to depend on direction once it has been accepted that the wave function rotates about a definite axis specified by the spin. For any pair of rectangular coordinates perpendicular to the spin axis, the uncertainty relations can be given a clear and unambiguous interpretation. They merely express the fact that the average internal kinetic energy of the packet has a fixed value $\hbar/2\tau$, while the average internal angular momentum of the packet has the fixed value $\hbar$; these two constraints imply a relation between position and velocity. The uncertainty relation (3.22) has its minimum value at $t = 0$ when all the internal energy is rotational kinetic energy. As the packet expands the rotational energy is converted to radial kinetic energy and the radius of the packet expands in such a way that the angular momentum remains constant. Indeed, Eqs. (3.20) and (3.21) show that the internal energy $\Delta p_x^2/2m = \hbar/4\tau$ remains constant; while it is the width $\Delta x$ alone that accounts for the "increase in uncertainty" with time in the relation (3.22). Thus, it appears that the uncertainty relations are merely reflections of the more fundamental fact that the *electron has a zero point energy $\hbar/2\tau$ associated with a zero point angular momentum of magnitude $\hbar$*. The magnitude of the zero point energy is not fixed by the PS theory, because $\tau$ is an undetermined parameter. However, the possible values of $\tau$ are certainly restricted by the requirement of consistency with the value for the angular momentum.

The uncertainty relation (3.22) also applies to a coordinate directed along the spin axis, but from our earlier discussion of internal energy, it should be clear that it cannot then be directly attributed to constraints on the angular momentum and kinetic energy of the electron. We shall have more to say about this later on.

A harmonic oscillator wave packet can be analyzed in the same way as a free particle wave packet. Messiah shows that the minimum wave packet for an isotropic oscillator with natural frequency $\omega_0$ has the form[7]

$$\psi = \frac{1}{(2\pi a^2)^{3/4}} \exp\left( \frac{i\boldsymbol{\sigma}_3 \mathbf{x} \cdot \langle\, \mathbf{v} \,\rangle}{\hbar} - \frac{(\mathbf{x} - \langle\, \mathbf{x} \,\rangle)^2}{4a^2} \right) , \tag{3.23}$$

with the origin at the center of force. The probability density of this packet has the Gaussian form (3.4), however, unlike the free particle case, it has a fixed width given by

$$a^2 = (1/3)\langle\, (\mathbf{x} - \langle\, \mathbf{x} \,\rangle)^2 \,\rangle = \hbar/2m\omega_0 \,. \tag{3.24}$$

The centroid and velocity of the packet satisfy the "classical" differential equation

$$\frac{d}{dt}\langle\, \mathbf{v} \,\rangle = \frac{d^2}{dt^2}\langle\, \mathbf{x} \,\rangle = -\omega_0^2 \langle\, \mathbf{x} \,\rangle \,. \tag{3.25}$$

These are well-known facts about an oscillator packet.

We learn something new when we look at the Pauli current and find that the local electron velocity is

$$\mathbf{v} = \langle\, \mathbf{v} \,\rangle = \boldsymbol{\omega} \times (\mathbf{x} - \langle\, \mathbf{x} \,\rangle) \,, \tag{3.26a}$$

where

$$\boldsymbol{\omega} = \mathbf{s}/ma^2 = \omega_0 \boldsymbol{\sigma}_3 \,. \tag{3.26b}$$

This tells us that the charge distribution of the oscillator packet is a "Gaussian ball" rotating rigidly about an axis through its centroid with a constant angular velocity determined by the natural frequency of the oscillator. The internal kinetic energy of this rotation is

$$\langle\, (1/2)m(\mathbf{v} - \langle\, \mathbf{v} \,\rangle)^2 \,\rangle = \langle\, (1/2)m[\boldsymbol{\omega} \times (\mathbf{x} - \langle\, \mathbf{x} \,\rangle)]^2 \,\rangle = (1/2)m\omega_0 \,. \tag{3.27}$$

As in the free particle case, the average angular momentum decomposes into an angular momentum of the centroid plus the intrinsic angular momentum $2\langle\, \mathbf{s} \,\rangle = \hbar\boldsymbol{\sigma}_3$;

$$\langle\, \mathbf{x} \times m\mathbf{v} \,\rangle = \langle\, \mathbf{x} \,\rangle \times \langle\, m\mathbf{v} \,\rangle + \hbar\boldsymbol{\sigma}_3 \,. \tag{3.28}$$

This tells us that the oscillator has a minimum angular momentum of magnitude $\hbar$ in the ground state, for which $\langle\, \mathbf{v} \,\rangle = 0$. It follows that the one-dimensional oscillator, so commonly studied in quantum mechanics, is a completely unrealistic model of electron motion, in spite of the mathematical separability of Schrödinger's equation for the oscillator into uncoupled one-dimensional equations.

The PS equation for a two-dimensional oscillator is the simplest model that can accomodate the electron's intrinsic angular momentum. In this case, the entire kinetic energy is given by (3.27).

According to the virial theorem, the time average of the kinetic energy is equal to the time average of the potential energy for a bound particle, so from (3.27) we get an average energy

$$\overline{\langle\, \mathbf{E} \,\rangle} = m\overline{\langle\, \mathbf{v} \,\rangle^2} = \hbar\omega_0 \,. \tag{3.29}$$

The important point here is that the ground-state energy can be entirely attributed to the kinetic and potential energy of a point particle circulating with angular momentum $\hbar$. According to the general equation (2.10), the three-dimensional oscillator differs from two-dimensional oscillator by an additional contribution to its kinetic energy from the term

$$\frac{1}{2}m\langle\, c^2\beta^2 \,\rangle = \left\langle\, \frac{(\mathbf{s} \cdot \boldsymbol{\nabla} \ln \rho)^2}{2m} \,\right\rangle = \left\langle\, \frac{[\mathbf{s} \cdot (\mathbf{x} - \langle\, \mathbf{x} \,\rangle)]^2}{2ma^4} \,\right\rangle = \frac{1}{4}\hbar\omega_0 \,.$$

By the virial theorem, this leads to an increase of the ground-state energy by the amount $(1/2)\hbar\omega_0$. As said before, the significance of the term (3.30) is problematic.

Uncertainty relations for the oscillator are easily derived in the same way that we derived them for the free particle, but they only confirm what we have already found out, namely, that the "uncertainty" can be attributed to the local circulation of the electron with a fixed average energy and angular momentum.

## IV. STATISTICAL INTERPRETATION OF QUANTUM MECHANICS

To sustain the interpretation of observables in PS theory which has been set forth in Secs. II and III, it appears necessary to suppose that the wave function represents a statistical ensemble in the sense of Gibbs. Arguments for the cogency and consistency of this general

interpretation of the wave function have been reviewed by Ballentine,[8] who emphasizes that it was the viewpoint adopted by Einstein.

If the PS wave function does indeed describe a statistical ensemble of particles it should also be possible to describe the ensemble by a phase space distribution function $f = f(\mathbf{p}, \mathbf{x}, t)$ which projects to the probability density $\rho = \rho(\mathbf{x}, t)$ according to the rule

$$\rho = \int d^3p\, f\,. \tag{4.1}$$

A general method for constructing such a distribution function from the Schrödinger wave function was devised by Wigner.[9] But results have been somewhat unsatisfactory, because it has never been clear just what condition should be imposed to produce a unique distribution function. The present paper changes the question drastically, first, because it identifies the Pauli current instead of the Schrödinger current as the probability current, and second, because it requires that the spin be introduced, not as a new degree of freedom, but as a constraint on the motion in phase space, a constraint which generalizes the constraint imposed by the uncertainty principle in Wigner's treatment.

Without entering into an extended discussion of the phase-space formulation of quantum mechanics, some insight into the interpretation of the PS theory may be gained by considering what it would require of the momentum and energy moments of the distribution function. The *ensemble average* momentum must produce the Pauli current

$$\rho\mathbf{v} = m^{-1}\int d^3p\, f\mathbf{p}\,. \tag{4.2}$$

To be consistent with the expression (A35) for kinetic energy density in the Pauli theory, the ensemble average kinetic energy must yield

$$(2m)^{-1}\int d^3p\, f\mathbf{p}^2 = \frac{1}{2}mp(\mathbf{v}^2 + \beta^2) + \boldsymbol{\nabla}\cdot\mathbf{q}\,. \tag{4.3}$$

Clearly, for the phase space formulation to give the average kinetic energy (2.10) required by the PS theory, the vector field $\mathbf{q}$ in (4.3) need not be identical with the quantity $\rho\mathbf{v}\times\mathbf{s}+\rho c\beta\mathbf{s}$ in (A35). Since only a finite number of average values can ever be measured experimentally, there will always be some leeway in the formulation of a phase space theory.

Equation (4.3) suggests that the *parameter $\beta$ describes energy fluctuations* in the ensemble by specifying an average deviation in the energy of ensemble members from the energy associated with the average momentum $m\mathbf{v}$. This interpretation for the parameter $\beta$ has been suggested for the Dirac theory, in which $\beta$ may include fluctuations due to pair creation.[10] If this is correct, a careful study of the role that $\beta$ plays in the theory should supply insight into the nature of quantum mechanical fluctuations.

The oft discussed question of the completeness of quantum theory can be attacked as the quite definite physical question: how much can be known about individual members of a "PS ensemble"? Our analysis in Secs. II and III suggests that several things might be claimed as "known" already. Although the Pauli current presumably describes only an ensemble average, our analysis of its properties strongly suggests that the ensemble consists of point particles, each with the definite charge $e$ and mass $m$. Since members of a Gibbsian ensemble are presumed not to interact, it is hard to see how the spinning of a free particle

wave packet might be accounted for unless each free particle in the ensemble has a tendency to circulate about a "guiding center." To explain the significance of this peculiar behavior and account for the average angular momentum $2\mathbf{s} = \hbar\boldsymbol{\sigma}_3$ and fluctuations around it is a major problem of quantum electron theory. Some speculations will be offered later on.

Every solution of the PS equation determines a PS ensemble, but there is reason to believe that some such ensembles specify more than can be known about a physical system, though they may be useful nevertheless. The energy density $\rho E$ obtained from (A37) for an energy eigenstate is a constant multiple $E$ of the probability density $\rho$ at every point of space and time, suggesting that every particle in the associated ensemble has the same energy. Similarly, for an eigenfunction of spin, the spin density $\rho\mathbf{s}$ is everywhere proportional to a constant vector $\mathbf{s}$, so a spin eigenfunction describes an ensemble of particles circulating about a common direction in space. In general, then, it appears that eigenfunctions in the PS theory can be identified with ensembles for which the corresponding eigenvalue is a common property of each member. However, the energies of atomic states have finite natural widths not determined by the sharp energy eigenvalues, so the ensembles defined by energy eigenfunctions are too narrow to fully represent the physical properties of atomic states, and a quantum theory of radiation had to be developed to rectify this. The point is that eigenfunctions are not the fundamental representatives of physical states they are sometimes supposed to be. Fully satisfactory criteria determining the ensemble which best describes a given physical system are yet to be developed.

Plane wave solutions of the PS equation are particularly suspect as representations of a free particle state, because they differ qualitatively from the wave packet solutions by failing to display the circulation of charge required of physical solutions by our general considerations.

In search of greater completeness or just a better understanding of the PS theory, we ask to what extent the established predictions of the theory can be generated from subensembles of the theory. Subensembles with $\beta = 0$ are of particular interest, since the contribution of $\beta$ to experimental predictions is problematic. In the specific solutions of the PS equation which we have already considered, it will be noticed that $\beta$ is constant in planes perpendicular to the spin vector. So we expect $\beta = 0$ only for two-dimensional solutions of the PS equation, which may be regarded as describing a subensemble of particles confined to a plane. It is readily verified that for planar solutions of Schrödinger's equation for hydrogen the radial wave equation is identical to the radial wave equation for the usual three-dimensional problem, so the energy eigenvalues are identical in both cases. The same is true of corresponding solutions to the Dirac equation.[11] However, the planar solutions naturally do not show the dependence on the magnetic quantum number found in the three-dimensional solutions. It has not been determined whether or not the correct magnetic dependence can be superimposed on the planar solutions without reintroducing $\beta$ and the full three-dimensional solutions.

By considering subensembles of the PS theory which contain only one particle we get something very like the Bohr theory. Certain expectation values of the PS theory can be duplicated by a one-particle model, but all information about fluctuations is lost. The Bohr theory is not as different from the PS theory as commonly supposed. We have seen how the major discrepancy between the Bohr and Schrödinger descriptions of the hydrogen ground state was eliminated by reexpressing spin as an orbital angular momentum. Let us look at the matter in another way.

For a "Bohr ensemble" of one particle in a ground state, Eq. (2.5) requires that the particle have the angular momentum

$$\langle\, \mathbf{r} \times \mathbf{p} \,\rangle = \mathbf{r} \times \mathbf{p} = \hbar \boldsymbol{\sigma}_3 \,. \tag{4.4}$$

If the electron is in a hydrogenlike atom, the ground state can be found by minimizing the energy

$$E = \mathbf{p}^2/2m + Ze^2/r$$

subject to the constraint (4.4). One finds a circular orbit with the expected Bohr radius and the same energy as predicted by both the Bohr and Schrödinger theories. This is similar to an old argument[12] in which the uncertainty relation is supposed to justify the constraint

$$rp = \hbar \,, \tag{4.5}$$

from which the same result follows, because (4.5) follows from (4.4). Note, however, that if (4.5) is based directly on the minimal uncertainty relation $\Delta x \Delta p = (1/2)\hbar$ by writing $r = \Delta x$ and $p = \Delta p$, the variables in (4.5) have a meaning different from the one we assigned to them and one has to "fudge" a factor of 1/2 Thus, the old argument lacks the precision of ours and fails to find the crucial relation between ground-state energy and ground-state angular momentum.

Our argument is applied to the isotropic oscillator by minimizing the energy

$$E = \mathbf{p}^2/2m + (1/2)m\omega_0^2\mathbf{r}^2 \,,$$

subject to the constraint (4.4). One finds $r^2 = \hbar/m\omega_0$ and energy $E = \hbar\omega_0$ in exact agreement with the prediction of the Schrödinger equation for a two-dimensional oscillator. Again we see how the zero-point energy is associated with a zero-point angular momentum.

Equation (4.4) or (4.5) can be interpreted as an expression of the uncertainty principle for a single particle. If the particle is confined to a region of dimension $r$, then, according to (4.5), it must have a momentum of magnitude $p$ at the least to maintain the required minimum angular momentum $\hbar$. If the region is diminished, then the momentum must be increased to maintain the required minimum angular momentum. Moreover, along directions orthogonal to $\boldsymbol{\sigma}_3$ the component of momentum is continually fluctuating in sign and magnitude as the particle circulates. This much is in general agreement with the conventional interpretation of the uncertainty relations. However, Eq. (4.4) suggests that the components of position and direction may be specified independently along the $\boldsymbol{\sigma}_3$ direction, so the uncertainty relations are not symmetrical in space. But, we should be cautious about such an inference, because we know that the symmetry is restored by the parameter $\beta$ which was eliminated along with fluctuations when we confined our attention to a single particle. Just the same, we can attach considerable credence to this interpretation of the uncertainty principle, because it conforms so well with established facts about the electron spin and magnetic moment.

It is natural to ask if there is some rigorous way to relate our one-particle "loop model" of the uncertainty principle to the "spinning wave packet" describing a complete PS ensemble. A possibility is suggested by considering effects of the electromagnetic vacuum field to be discussed in Sec. VI. Vacuum fluctuations in resonance with the loop would be expected

to produce random small displacements of the loop, producing a shift in its average energy without altering its angular momentum. The result would be that the loop undergoes a random walk with consequent uncertainty in the position of the particle. Properties of the spinning packet in Sec. III, especially an expression like (3.7) for the mean square displacement of the loop, are very like what one would expect from a stochastic description of such a process. It would be likely, then, that the problematic parameter is a general consequence of vacuum fluctuations, and, along with it, the general "smearing out" of electron orbits so familiar in the Schrödinger theory. However, for reasons given in Sec. V, it seems unlikely that such an explanation can be sustained without some modification of quantum electrodynamics.

## V. PARTICLES AND PERIODICITIES

The statistical interpretation of quantum mechanics adopted in this paper flatly contradicts the orthodox Copenhagen interpretation, though it is not without supporters in the physics community. The orthodox Copenhagen interpretation and its many variations has been discussed and criticized at length in the literature.[13,14] We merely wish to point out that it has *no* advantages over the statistical interpretation. Indeed, we achieve a more complete interpretation of details in the PS electron theory if we do not introduce the paradoxical concepts of duality and complementarity.

The uncertainty relation $\Delta x \Delta p_x \geq (1/2)\hbar$ is said to be a consequence of wave-particle duality. Let us analyze its meaning from our statistical viewpoint. First consider the experimental significance of the terms. Experimentally, the term $\Delta x$ refers to the scatter of independent position measurements on electrons in similar physical states. The term $\Delta p_x$ refers to the scatter in momentum measurements (successive pairs of position measurements). A statistical ensemble is determined by the specification of conditions for the experimental preparation of similar physical states. A measurement is to be regarded as a sampling of particles in the ensemble. The theoretical value for $\Delta x$ describes limits on the accuracy to which the position of electron is predicted. The experimental value of $\Delta x$ describes limits on the accuracy to which the position of an electron is controlled. The "uncertainty principle" predicts that it is impossible to prepare an experimental ensemble for which $\Delta x \Delta p_x \geq (1/2)\hbar$. There is no limitation here on the accuracy with which position and momentum can be simultaneously measured, only a limitation on the degree to which they can be simultaneously controlled. The fact that position measurement cannot be made with sufficient accuracy to test the uncertainty principle directly is quite beside the point. The meaning of a measurement should not be confused with its feasibility.

Now we turn to the theoretical significance of the uncertainty relations, which is determined by the requirement of consistency with other concepts in the theory. This matter has been considered in previous sections where it was argued that the uncertainty relations are merely a by-product of a more fundamental property of electrons, namely, a minimum angular momentum manifested experimentally as a magnetic moment. Of course, this argument does not make sense unless the electron is regarded as a particle, so it is inconsistent with the notion of wave-particle duality. For this reason, the Copenhagen viewpoint appears to be quite unable to reconcile the uncertainty relations with the fact of spin, though the dependence of both on Planck's constant cries out for some explanation. Indeed, the

duality principle would prevent us from identifying the Pauli current unambiguously as a local flow of electrons and so makes it impossible to interpret the terms in our mathematical analysis of the local structure of the PS theory. Even the uncertainty relation, which is supposed to be explained by duality, becomes ambiguous when it is not clear whether $\Delta x$ refers to measurements on particles or waves. Judged by its ability to interpret the details of mathematical formalism in quantum theory, the Copenhagen viewpoint is profoundly deficient if not demonstrably wrong.

We cannot end this discussion without speaking to the chief argument for the necessity of wave-particle duality, namely the claim that diffraction cannot be understood except as interference of a wave by itself. This claim is false. As the late Alfred Landé has repeatedly stressed, diffraction can be explained by the concept of quantized momentum exchange without reference to waves.[14] His review of the original work by Duane[15] may have left some readers with the impression that the explanation has some special connection with old quantum theory. But the momentum exchange in diffraction is calculated quite directly in modern many body theory without appeal to the idea of interference.[16] The discrete spectrum of vibrational states in a crystal is determined by symmetry properties of the crystal wave function, and the Bragg diffraction formula is determined by the allowed transitions between those states. Notice that no assumptions about the nature of the diffracted object or the mechanism of momentum exchange are required. Crystal diffraction patterns for light and electrons are similar, not because they are both behaving as waves, but because the spectrum of momentum that they exchange in diffraction is determined by the symmetry properties of the crystal alone.

To say that an electron is a particle and not a wave is not to deny de Broglie's hypothesis associated the there is a periodicity associated with electron motion. The properties of periodicity in motion and extension in space associated with the wave concept are conceptually independent, so one can be ascribed to a physical system without the other. The PS wave function has both properties, but the statistical interpretation requires that the extension in space is a property of the ensemble as a whole, whereas the periodicity may reflect properties of individual members of the ensemble. Indeed, our analysis of PS ensembles seems to support the idea that spin reflects a tendency of individual electrons to execute circular motion with quantized angular momentum even in the absence of external interactions. This idea has the great virtue of integrating the concepts of intrinsic spin and periodicity into a single model of electron motion, so it seems to be worth pushing as far as it will go.

In our analysis of the free particle wave packet, we found that the electron motion can be separated in two independent parts: an "intrinsic motion" (determining the electron's magnetic moment) and an "extrinsic motion" of the guiding center. These two motions are coupled when the electron is bound. Equation (3.26b) shows that in an isotropic oscillator the frequency of the intrinsic motion is equal to the natural frequency of the oscillator. This suggests that it may be possible to explain the quantum conditions for stationary states as conditions of coherence between intrinsic and extrinsic motions, which, in turn, calls for an explanation of why such a coherence should be associated with stationary states. Speculation on this point will be made in Sec. VI.

## VI. QUANTIZATION, FLUCTUATION, AND RADIATION

The statistical interpretation raises some serious questions about the completeness and self-consistency of quantum mechanics to be discussed in this section. We begin with the problem of reconciling the statistical interpretation of the wave function with the facts about stationary states. Our analysis leads to the conclusion that all stationary electron states produce fluctuating electromagnetic fields. Reasons are given for believing that such fields are responsible for Van der Waals forces. To see how the idea might be worked into quantum electrodynamics, we analyze the status of quantization postulates in conventional theory and speculate on the possibility of improvements.

There is little room to doubt that the mathematical characterization of stationary energy states in the PS theory is fundamentally correct. Among other things, it provides us with a quantitative description of atomic and molecular spectra which has been experimentally verified in great detail. The PS theory predicts that energy eigenstates are distinguished by electron currents which produce static magnetic fields. Complex spectra reveal the effects of such static fields in many ways. Indeed, the spectra cannot be interpreted without assuming their existence. It seems safe, therefore, to regard the existence of stationary atomic currents as an established fact. A coherent interpretation of quantum mechanics should be expected to reconcile this fact with other facts in the theory. This presents us with a series of problems.

The first problem is to reconcile the atomicity of electron charge with the fact that the stationary electron currents are continuously distributed in space. Schrödinger was acutely aware of this problem from the beginning. He originally assumed that $e\psi^\dagger\psi$ represents a continuous distribution of charge with a stationary flow given by the Schrödinger current. However, he soon realized that this interpretation faces insuperable difficulties, not the least of which is the complete absence of any evidence for electrostatic interaction between different elements of a charge distribution.

Born's statistical interpretation of $\psi^\dagger\psi$ as a probability density avoids the problem of electrostatic self-interaction, but to reconcile this with the fact of stationary currents associated with individual electrons in individual atoms, it appears necessary to assume that a stationary electron current accurately represents a *temporal average* motion of a single point charge. However, this interpretation has implications that go beyond the PS theory. In agreement with the PS theory, it predicts that a circulating electron will produce a magnetic field. But a circulating point charge will also produce a fluctuating electric field, and such a field does not appear in the PS theory. Therefore, without denying that quantum mechanics is correct as far as it goes, our statistical interpretation implies that quantum mechanics is *incomplete*, because it does not associate fluctuating electric fields with stationary states.

The argument just presented assumes that electron is a point particle with a continuous trajectory, although it is not assumed that quantum mechanics can or should give a complete description of such a trajectory. Fortunately, we need not discuss whether this is a "naive classical picture," because it presents us an experimental prediction that should enable us to distinguish it from alternatives that deny that electrons have trajectories. It should be possible to detect electric fields that fluctuate with the periods of electrons in stationary states. Indeed, extensive experimental evidence for such fluctuating fields may exist already, primarily in measurements of long-range molecular forces known as Van der

Waals forces. Let us briefly review the modern explanation of Van der Waals forces due chiefly to London.[17]

For present purposes, it will be sufficient to consider the interaction between two widely separated hydrogen atoms in the ground state. From standard formulations of Schrödinger theory one might conclude that the electromagnetic force of one atom on the other is zero, because the $s$-state charge distribution is spherically symmetric. However, if the electron is a particle as demanded by the statistical interpretation, the atom has an instantaneous dipole moment. The dipole moment will fluctuate with fluctuations in the position of the electron, and it must have a time average of nearly zero in accordance with experimental measurements. Nevertheless, the instantaneous dipole moment of one atom will tend to polarize the other atom and establish a phase relation between their fluctuations so that the average interaction does not vanish. The strength of the interaction is calculated by second order perturbation theory.[18] The result depends on the probability of optical transitions between the ground state and higher level stationary states.

This brief account of the standard theory of Van der Waals forces is intended to emphasize two points of special interest here. First, the standard theory does not make sense unless electrons are regarded as particles producing fluctuating dipole moments. It seems strange that this point is not often made in discussions of the interpretation of quantum mechanics. True enough, perturbation calculations for Van der Waals forces can be carried out without commitment to any interpretation, but some interpretation is essential to tell us what perturbing forces to introduce and why.

The second point to be noted is that the standard theory does not give a well-defined account of the fluctuations producing the Van der Waals forces. Here, it seems, there is a need for some fundamental improvement and extension of the theory. Our kinetic interpretation of electron spin brings up some new possibilities. Ordinarily, the fluctuations in position of an electron in an $s$ state are supposed to be random, and it is hard to understand why they should produce the optical transitions associated with the Van der Waals forces. But the kinetic interpretation of spin implies that position fluctuations have a systematic component on which random fluctuations are possibly superimposed. It requires that the electron circulate with an average angular momentum of magnitude $\hbar$. This will produce a rapidly rotating electric dipole. Considering an electron in a Bohr orbit, we estimate the period of the resulting fluctuating field as $10^{-21}$ sec. A periodic field like this has greater potential for inducing correlations between dipole moments and driving optical transitions than does a random field. But, more important, our model suggests that *the strength of the Van der Waals force depends on the orientation of spin*, and so on the presence of magnetic fields. An effect of this kind should be observable, but this is not the place to pursue the issue.

Now let us turn to some problems raised by our considerations. We must ask why the fluctuating electromagnetic field of a circulating electron in a stationary state does not carry away the electron's energy as radiation. Ironically, textbooks seldom fail to find fault with the Bohr theory for not explaining why stationary states are radiationless, yet the issue is never raised in connection with the Schrödinger theory. Modern quantum mechanics assumes that solutions of the time-independent PS equation describe radiationless states, with no more explanation than an assertion that the electron currents do not radiate because they are static. But we have seen that this is difficult to reconcile with the atomicity of electron charge, so let us speculate on the possibility of resolving this problem by explaining

radiationless motion.

Since stationary states are characterized by quantum conditions, one might surmise that an equation of motion that correctly describes the coupling of an electron to its own field has the property that radiation vanishes when the quantum conditions are satisfied. The observation made at the end of the last question that quantum conditions might be the result of coherence between internal and external motions of an electron suggests a mechanism for explaining radiationless electron motion. If the two motions are regarded as independent sources, then an accelerating electron undergoing periodic motion is the source of two periodic fields. If the two radiation fields are coherent they may interfere destructively, which would reduce the net radiation if not eliminate it entirely. This particular mechanism for producing radiationless motion is not so important as the general idea that one should look for modifications of electrodynamics that allow time-dependent fields without radiation and examine new possibilities opened up by the idea that spin is a dynamical property of electron motion.

If there is truth to these suggestions, then we should be able to explain why the free particle wave packet spins. The free electron is not entirely free, because it is coupled to its own electromagnetic field. If the electron mass is the energy of self-interaction, as widely believed, it may be that the spin is an angular momentum of the self-interaction. If so, the usual coupling of the electron to its field must be modified so that the radiative reaction does not reduce to zero for a free particle but to a finite value determined by the spin. This would entail a relation of the electron's self-energy to its self-angular momentum, so it should have bearing on the problem of self-energy divergences that plague current theory.

Spin is already known to play a crucial role in self-energy calculations of conventional quantum electrodynamics. Divergent fluctuations of electron kinetic energy induced by the electromagnetic vacuum field are exactly cancelled by the oscillatory motion that produces the spin.[19] If this is less than a coincidence, it points to an intimate relation of spin to the vacuum fluctuations. If our dynamical interpretation of spin is correct, then the circulatory motion of any electron, bound or free, will produce an oscillating electric field as well as a magnetic moment. It would be a great conceptual economy if it could be shown that this fluctuating field performs the functions assigned to the vacuum field in conventional theory, for then the strenuous expedient of second quantization would be unnecessary.

The only indubitable physical consequences of second quantization arise from the action of the electromagnetic vacuum field,[20] which is held responsible for spontaneous emission, the Lamb shift and the anomolous magnetic moments of electrons and muons. One problem with second quantization is that it is introduced by assumptions which are logically independent of those for "first quantization." In conventional quantum electrodynamics Planck's constant is introduced twice by independent assumptions; first, by assuming the Dirac equation or its nonrelativistic approximation the PS equation, and second, by assuming the canonical commutation relations for electromagnetic field variables. These two assumptions are related by no more than a qualitative argument to the effect that quantization of the electromagnetic field is necessary for the sake of consistency with the uncertainty principle.[21] Of course, no one believes that Planck's constant should appear in the "ultimate theory" any more than once, so it is reasonable to look for alternative explanations of "vacuum field effects."

Our proposed explanation of the origin of spin and quantization requires that free and bound electrons are sources of fluctuating fields which we would like to identify as the "true

electromagnetic vacuum field." Let us compare this general picture of the vacuum field with the one given by second quantization.

Second quantization rigorously determines the properties of the electromagnetic vacuum field and admits no possibility for change. This vacuum field has infinite energy and no sources. In contrast, the theory of the electromagnetic vacuum field suggested here is incomplete and promises to be a subject of research for a long time. This vacuum field has finite energy and is rooted in matter, though under some conditions it may be well approximated by the power spectrum of the infinite vacuum field and so reproduce the well-verified predictions of quantum electrodynamics.

By the way, it is interesting to note the similarity of the model of the electromagnetic vacuum field suggested here to the original model suggested by Slater in 1924.[22] In order to account for the connection between "spontaneous" and "induced" radiation made explicit by Einstein's analysis of black body radiation, Slater supposed that atoms communicate with one another by means of a "virtual radiation field." Slater's idea played a key role in the development of matrix mechanics,[23] leading ultimately to quantization of the electromagnetic field, with its model of the vacuum field in which, ironically, the original idea of "communication between atoms" is completely missing.

There ought to be ample opportunity to distinguish between models which hold that the electromagnetic vacuum field has its origin in matter, and the model of quantum electrodynamics which holds that the vacuum field is wholly independent of the distribution of matter. In this connection, an alleged success of quantum electrodynamics may be mentioned.

By quantizing the free electromagnetic field between parallel plates a temperature independent attractive force is found (Casimir effect),[24] with magnitude in agreement with experiment.[25] The force is completely determined by the energy density of the vacuum field between the plates. Now quantum electrodynamics provides no justification for this application of macroscopic boundary conditions to the vacuum field. Second quantization introduces the vacuum field with no reference to matter. From the viewpoint advanced here, however, the use of boundary conditions may be regarded simply as a convenient way to express the fact that the plates themselves are the source of the vacuum field. This example shows that vacuum field contributes to molecular forces. Or better, the vacuum field is a manifestation of molecular fields. Each molecule is the source of a fluctuating electromagnetic field. At long ranges this field produces an attractive force between molecules which we have identified with the Van der Waals force.[26] The temperature independent part of the composite long range fluctuating field due to all molecules is to be associated with the vacuum field.

Our discussion of quantization would be incomplete without some account of the Pauli exclusion principle. The Pauli principle puts severe constraints on the allowed motion of electrons, apparently without the intermediary of a force. But note that it applies only to stationary states, that is, to a system of identical particles in radiationless motion. So the status of the Pauli principle is similar to that of the quantum conditions; it is a condition for radiationless motion. Presumably the interaction between electrons is such that no two of them can be in the same stationary state without radiating. This proposal for explaining the Pauli principle obviously agrees with our account of spin and would automatically give the correct relation between spin and statistics for electrons.

Conventional quantum electrodynamics offers no explanation at all for the Pauli principle,

though certain inconsistencies are shown to result if it is not assumed. Usually, the Pauli principle is formally introduced into the theory by assuming anticommutation relations for operators referring to identical particles. But this would seem to unnecessarily limit the theory to stationary states from the beginning, if our guess about the origin of the Pauli principle is correct. The operator formalism of second quantization may be a great mathematical convenience. On the other hand, it could be a conceptual straightjacket.

One final point: it seems to be widely believed that the Pauli principle is closely associated with the "principle of indistinguishability of identical particles." It is true that both principles are needed in many-particle theory, and both apply only to systems of identical particles. Nevertheless, they are quite unrelated. The Pauli principle is not a statistical principle; it is a dynamical principle, because it limits the allowed motions of particles. In contrast, the principle of indistinguishability has nothing to do with dynamics; indeed, it can be justified without appeal to any special features of quantum theory.[27] Common usage of the term "quantum statistics" indicates a widespread confusion of dynamics with statistics.

## VII. COMPLEX NUMBERS IN QUANTUM MECHANICS

The prominent role of complex numbers in quantum mechanics is usually taken for granted and justified by its implications. I would like to call attention to an important *fact* which shows that a deeper analysis is possible with potentially important physical implications. That fact is that *the unit imaginary in the Schroedinger wave function is the generator of rotations about the axis of spin.* (See the Appendix for a mathematical formulation.)

Let us be clear about the ontological status of this purported fact. It is not an interpretation arbitrarily imposed on the Schrödinger wave function by external considerations as is the case with wave-particle duality. Rather it comes from requiring that quantum mechanics be internally consistent, specifically, that the interpretation as well as the equations of Schrödinger theory be derived from the Dirac theory.[28]

The consistency argument establishes the relation of spin to the unit imaginary in the Schrödinger wave function for electrons but not for pions, which are also commonly described by complex wave functions. Therefore, some new facts and ideas will be required to extend what has been established in electron theory to a coherent physical interpretation of all complex wave functions in quantum theory. By way of suggestion, let us speculate briefly on some possibilities.

Two facts of general significance are available to guide our speculations: First, as already mentioned, the unit imaginary in the electron wave function has a definite geometrical interpretation by virtue of its relation to spin. Second, distinct wave functions in quantum mechanics are related by the group structure of elementary particle theory. By combining these facts in a unique model, we may be able to establish a geometrical interpretation in spacetime of all wave functions and dynamical transformation groups relating them. This is obviously a tall order, so the best that can be done here is to give some hint as to how one might try to fulfill it.

This paper has been concerned primarily with nonrelativistic quantum mechanics, but the basic facts about spin and its relation to complex numbers come from the Dirac theory

which, with some modification, is presumed to apply to all fermions. This helps us respond to the natural question: if Schrödinger's equation (or the Klein-Gordon equation, for that matter) is a reasonable equation of motion for pions in some approximation, as seems to be the case, what geometrical interpretation can be given to the imaginary unit in the complex pion wave function, considering the fact that pions are spinless particles? A possible answer is suggested by regarding the pion as a nucleon-antinucleon or quark-antiquark bound state. Although the spin of the pion is zero, the spins of its constituents may well have definite directions. If the spin vector of the fermion constituent is in the direction $\boldsymbol{\sigma}_3$, then the spin vector of the antifermion must have direction $-\boldsymbol{\sigma}_3$ to produce the spin zero bound state. It thus seems reasonable to suppose that the unit imaginary in the pion wave function is $i\boldsymbol{\sigma}_3$, where $\boldsymbol{\sigma}_3$ describes the common spin axis of its constituents.

This much being granted, we find that we must go further. The vector $\boldsymbol{\sigma}_3$ imbedded in the pion wave function is evidently a constant of motion, yet it must certainly have different directions for different pions. It is natural, then, to identify these directions with the states of pions in isospace, so that the imaginary unit in each pion wave function is also a generator of rotations in isospace. The number of degrees of freedom available is exactly right. Thus we arrive at the possibility of a theory relating spin and isospin and so associating isospace with spacetime. A generalization of the Dirac theory along these lines was suggested some time ago.[29] The particular model is not so important as the basic idea for relating "internal symmetries" of elementary particle theory to structural properties of the Dirac equation.

There is another possibility for establishing a connection with "internal symmetries" for which a stronger case can be made. Our formulation of the PS equation in the appendix identifies a bivector $i\boldsymbol{\sigma}_3$ as the generator of electromagnetic gauge transformations. This is to be contrasted with conventional formulations in which the generator appears as an imaginary unit in some abstract space unrelated to physical space. The difference between these formulations has major implications for recent "gauge theories" which generalize the electromagnetic gauge group to get a unified theory of weak and electromagnetic interactions. The generator $i\boldsymbol{\sigma}_3$ already has a definite relation to physical space, so we can hardly accept less from other generators of the extended group. Moreover, the bivectors $i\boldsymbol{\sigma}_1$, and $i\boldsymbol{\sigma}_2$ are already available to complete the set of generators for an SU(2) gauge group as required by the gauge theories. This group automatically has an intimate connection with spin, just as the " V-A theory" of weak interaction coupling would lead one to expect. Of course, a unified theory of weak and electromagnetic interactions must be developed in connection with the Dirac theory rather than the PS theory. This is not the place to pursue it. But it can be pointed out that the elements of such a theory are already available in Ref. 29; one need only reinterpret the isospin transformations given there as gauge transformations and introduce interactions by the usual "Yang-Mills trick."

## VIII. SUMMARY

The argument in this paper is based on the observation that the conventional interpretation of the Schrödinger theory as describing an electron without spin is logically inconsistent with the Pauli and Dirac theories. Consistency requires that the Schrödinger equation describes an electron in an eigenstate of spin. Thus the Schrödinger equation must be regarded

as identical to the Pauli equation in the absence of magnetic fields. This alters the conventional interpretation of Schrödinger theory significantly. In the first place, it follows that the Pauli current $\rho\mathbf{v}$ rather than the Schrödinger $\rho\mathbf{u}$ must be identified as the probability current in the Schrödinger theory. A complete interpretation of the theory depends crucially on how these currents are associated with momentum and energy.

If $m\rho\mathbf{u}$ is identified as momentum density, as is (tacitly) done in conventional expressions for observables, then a term proportional to $\langle\, \mathbf{B}\cdot\mathbf{s}\,\rangle$ appears in the expression for energy in the Pauli theory, and this term is interpreted as the *potential energy* of an intrinsic electron magnetic moment in a magnetic field $\mathbf{B}$. Thus, the spin and magnetic moment are interpreted as properties of *electron structure*. On the other hand, if $m\rho\mathbf{v}$ is interpreted as momentum density, then the $\langle\, \mathbf{B}\cdot\mathbf{s}\,\rangle$ term is absorbed in the expression for *kinetic energy*. Thus, the spin and magnetic moment must be interpreted as *dynamical properties* of electron motion, and the theory is consistent with a model of the electron as a structureless point charge. Furthermore, the intrinsic spin of magnitude $(1/2)\hbar$ can be interpreted as an orbital angular momentum of magnitude $\hbar$. In this connection it is interesting, but not essential to notice that the factor $i\hbar$ in the Schrödinger equation describes both magnitude and direction of the unit of angular momentum, with $i$ representing the bivector generator of rotations in the plane orthogonal to the axis of quantization determining the "spin eigenstate." In this sense we can say that spin was already represented in the original Schrödinger equation.

Let us distinguish the two different interpretations of spin in the PS theory by the adjectives "structural" and "kinetic," respectively. We will not have an unambiguous interpretation of quantum theory until we can decide between these two possibilities. The structural interpretation will not submit to further analysis without introducing some model for the supposed structure of an electron, and we have no reliable guide for such a model. In contrast, the kinetic interpretation promises to be more complete, because no unknown structural properties need be presumed.

We have seen that even when there are magnetic interactions the kinetic energy consists of two terms: $(1/2)m\langle\, \mathbf{v}^2\,\rangle$ and $(1/2)mc^2\langle\, \beta^2\,\rangle$. The interpretation of the first term is obvious. The second term could also be given a kinetic interpretation if $\beta$ could be associated with fluctuations in the local velocity $\mathbf{v}$, but this would require some modification or extension of the conventional theory. For an electron bound or free, in a ground state, the first term contributes 2/3 of the ground-state energy, and all of this can be regarded as kinetic energy associated with a ground-state orbital angular momentum of magnitude $\hbar$. This leads us to a new interpretation of the Heisenberg uncertainty relations: The variance $\Delta p_x$ in momentum is determined by the groundstate kinetic energy associated with the ground-state orbital angular momentum. The correlation of $\Delta p_x$ with the variance $\Delta x$ in position is a consequence of the correlation between position and momentum determined by the ground-state angular momentum. Thus, the uncertainty relations are explained by interpreting electron spin as a minimum orbital angular momentum.

The kinetic interpretation of the uncertainty relations is imperfect, because the significance of $\beta$ is still obscure. However, $\beta$ is proportional to the divergence of the spin density, so all contributions to the ground-state energy depend on the spin. This dependence of the ground state on spin has gone unnoticed in conventional interpretations of quantum theory, because the necessary relation of the Schrödinger equation to spin has been overlooked. For this reason, the necessary relation of spin to the uncertainty principle has been overlooked

as well.

The problem of supplying a kinetic interpretation would be solved if the statistical interpretation of the PS theory could be brought to its logical conclusion as a stochastic theory of motion of a point particle. This program faces the difficult problem of explaining why stationary states are radiationless. It also presents new possibilities, such as a spin-dependence of Van der Waals arising from fluctuating fields associated with bound states, and connections with vacuum fluctuations of the electromagnetic field.

## APPENDIX: PAULI-SCHRÖDINGER THEORY

This Appendix summarizes the basic assumptions of the PS electron theory and, with a few variations, some general results from Refs. 2 and 3 which have important bearing on the physical interpretation of the theory. The relation of the PS theory to the Dirac theory is discussed in Ref. 3.

We formulate the PS theory in terms of the *geometric algebra* used in Refs. 2 and 3. This language has the advantage of making the geometrical features of the Pauli algebra and the Pauli theory quite explicit. Our notation is intended to make the correspondence with the conventional language as obvious as possible. Pauli matrices $\sigma_1$, $\sigma_2$, $\sigma_3$ in the conventional language correspond to vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$ in *geometric algebra*. The Pauli algebra is a representation of the *geometric algebra* by $2 \times 2$ matrices, but the use of matrices only obscures the geometric content of the algebra. To appreciate the geometrical significance of the language it is essential to understand that the products $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$, $\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1$, are bivectors (which specify the directions of planes in space, just as the vectors $\boldsymbol{\sigma}_k$ specify directions of lines); moreover, the product $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i$ is the *unit right-handed* pseudoscalar for space. It is essential to distinguish between the unit pseudoscalar $i$ in geometric algebra and the unit imaginary $i'$ in the Pauli matrix algebra which does not have a unique geometric significance. It is implicit in the relations below that the imaginary $i'$ in the conventional theory sometimes corresponds to the pseudoscalar $i$ and other times to the bivector $i\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, depending on how it is used in the theory. The interpretation of geometric algebra is fully discussed in Ref. 4, while its correspondence with the Pauli algebra is explained in Refs. 2 and 3.

In spite of its unconventional form, our formulation of the PS electron theory is mathematically isomorphic to the conventional one, so its physical consequences cannot be different. However, it does eliminate redundancy in the conventional language and make the "hidden" geometrical properties of complex numbers in quantum mechanics explicit. Redundancy is eliminated by integrating the algebras of vectors and spinors into a unified mathematical system. This should make the subject simpler and clearer. On the other hand, geometric algebra reveals that the unit imaginary in the Schroedinger electron theory must be interpreted as a bivector $i\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, and a full understanding of why this should be so may well lead to important new physical results, as suggested in Sec. VII.

The correspondence of our expressions to those in the conventional theory is given below, but to understand all details of this Appendix, at least a passing acquaintance with Refs. 2–4 is probably necessary.

## WAVE FUNCTION AND SPIN

Let $\Psi$ denote the two-component spinor wave function in conventional theory which *uniquely* corresponds to our "quaternion-valued" wave function $\psi = \psi(\mathbf{x}, t)$. The *probability density* $\rho = \rho(\mathbf{x}, t)$ is determined from either wave function by

$$\rho = |\,\psi\,|^2 = \psi\psi^\dagger = \psi^\dagger\psi = \Psi^\dagger\Psi\,. \tag{A1}$$

The *spin* $\mathbf{s} = \mathbf{s}(\mathbf{x}, t)$ is a vector field determined from by

$$\rho\mathbf{s} = (1/2)\hbar\psi\boldsymbol{\sigma}_3\psi^\dagger\,. \tag{A2}$$

This is related to the conventional expression for components of spin by

$$\rho\mathbf{s}\cdot\boldsymbol{\sigma}_k = (1/2)\hbar(\psi\boldsymbol{\sigma}_3\psi^\dagger\boldsymbol{\sigma}_k)_0 = (1/2)\hbar\Psi^\dagger\boldsymbol{\sigma}_k\Psi\,, \tag{A3}$$

where the subscript zero calls for the "0 vector" or "scalar" part of the quantity in parentheses.

The wave function $\psi$ cannot be directly measured in any experiment. At best it can be inferred from measurements of "observables" such as the density $\rho$ and the spin $\mathbf{s}$. From (A1) it follows that wave functions $\psi$ and $\psi'$ related by

$$\psi = \psi' U \tag{A4}$$

have the same density if

$$UU^\dagger = 1\,. \tag{A5}$$

Substituting (A4) into (A2), we get

$$\psi\boldsymbol{\sigma}_3\psi^\dagger = \psi'\boldsymbol{\sigma}'_3\psi'^\dagger\,, \tag{A6}$$

where

$$\boldsymbol{\sigma}'_3 = U\boldsymbol{\sigma}_3U^\dagger\,. \tag{A7}$$

From (A6) we can conclude that $\psi$ and $\psi'$ determine the same spin if $\boldsymbol{\sigma}'_3 = \boldsymbol{\sigma}_3$, in which case it follows from (A7) that

$$U = e^{i\boldsymbol{\sigma}_3\theta}\,, \tag{A8}$$

where $\theta$ is any scalar "phase angle." Thus, in the Pauli theory, two wave functions with the same spin density differ at most by a "phase factor" of the general form (A8).

Specification of the unit vector $\boldsymbol{\sigma}_3$ in (A2) is equivalent, in the conventional formulation of quantum theory, to specification of a *direction* (or *axis*) of *quantization*. With an appropriate choice of wave function, this direction is entirely at our disposal. Any rotation of a quantization axis $\boldsymbol{\sigma}_3$ into a new axis $\boldsymbol{\sigma}'_3$ can be expressed by an equation of the form (A7) where $U$ is independent of $\mathbf{x}$ and $t$. But this must be accompanied by a transformation of the wave function of the form (A4). In this case, Eq. (A6) shows that $\psi$ and $\psi'$ are merely different descriptions of the same physical system relative to different quantization axes.

An electron with wave function $\psi$ is said to be in an *eigenstate* of *spin* if its spin $\mathbf{s}$ is constant, that is, if $\mathbf{s}$ is a vector field independent of $\mathbf{x}$ and $t$. In this case, the quantization axis can be chosen to coincide with $\mathbf{s}$ and we write

$$\mathbf{s} = (1/2)\hbar\boldsymbol{\sigma}_3 \,. \tag{A9}$$

Then it follows from (A2) that the wave function has the form:

$$\psi = \rho^{1/2} e^{i\boldsymbol{\sigma}_3\theta} \,. \tag{A10}$$

This is equivalent to the conventional Schrödinger wave function, the only difference being that the unit imaginary $i\boldsymbol{\sigma}_3$ here has a clear physical interpretation; it is the generator of rotations in the plane orthogonal to the spin vector $\mathbf{s}$. This interpretation of the unit imaginary in the Schrödinger theory is no arbitrary assumption; it is an indisputable consequence of the fact that the Schrödinger equation for an electron must be a special case of the Pauli equation.

Any Pauli wave function can be written in the form

$$\psi = \rho^{1/2} U e^{i\boldsymbol{\sigma}_3\theta} = \rho^{1/2} e^{2i\mathbf{s}\theta/\hbar} U \,, \tag{A11}$$

where $UU^\dagger = 1$ and

$$\mathbf{s} = (\hbar/2) U\boldsymbol{\sigma}_3 U^\dagger \,. \tag{A12}$$

These equations describe an electron in an eigenstate of spin when $U$ is independent of $\mathbf{x}$ and $t$, and they obviously reduce to (A9) and (A10) when $U = 1$.

Relative to a specified quantization axis, a wave function can be expressed as a sum

$$\psi = \psi_+ + \psi_- \tag{A13}$$

of a "spin-up" wave function $\psi_+$ and a "spin-down" wave function $\psi_-$ defined by the conditions

$$\psi_\pm \boldsymbol{\sigma}_3 = \pm\boldsymbol{\sigma}_3 \psi_\pm \,. \tag{A14}$$

This corresponds to spin up (down) states $\Psi_\pm$, defined in conventional language by

$$\sigma_3 \Psi_\pm = \pm\Psi_\pm \,.$$

The up (down) components of the wave function can be written in the standard form

$$\begin{aligned} \psi_+ &= \alpha_+ e^{i\boldsymbol{\sigma}_3\theta_+} \,, \\ \psi_- &= \alpha_- e^{i\boldsymbol{\sigma}_3\theta_-}(-i\boldsymbol{\sigma}_2) \,, \end{aligned} \tag{A15}$$

where $\alpha_\pm$ and $\theta_\pm$ are scalars. Observing that

$$\psi_+\psi_-^\dagger = \alpha_+\alpha_- e^{i\boldsymbol{\sigma}_3(\theta_+ + \theta_-)}(i\boldsymbol{\sigma}_2) = -\psi_-\psi_+^\dagger \,, \tag{A16}$$

we find that the probability density is given by

$$\psi\psi^\dagger = |\,\psi_+\,|^2 + |\,\psi_-\,|^2 = \alpha_+^2 + \alpha_-^2 \,, \tag{A17}$$

while the spin density is given by

$$\begin{aligned}\psi\boldsymbol{\sigma}_3\psi^\dagger &= (\mid\psi_+\mid^2 - \mid\psi_-\mid^2)\boldsymbol{\sigma}_3 + (\psi_-\psi_+^\dagger - \psi_+\psi_-^\dagger)\boldsymbol{\sigma}_3\,,\\ &= (\alpha_+^2 - \alpha_-^2)\boldsymbol{\sigma}_3 + 2\alpha_+\alpha_- e^{i\boldsymbol{\sigma}_3(\theta_+ + \theta_-)}\boldsymbol{\sigma}_1\,,\end{aligned} \tag{A18}$$

Equations (A17) and (A18) contain the information commonly expressed in terms of a "density matrix" for a spin-$\frac{1}{2}$ particle. The quantities $\alpha_\pm^2 = \mid\psi_\pm\mid^2$ are interpreted as the relative probabilities for observing an electron in a spin-up (spin-down) eigenstate.

## OPERATORS AND CURRENTS

"Momentum operators" $\hat{p}_k$ are commonly defined by

$$\hat{p}_k\Psi \equiv\ -[i'\hbar\partial_k + (e/c)A_k]\psi\,.$$

where $\partial_k = \boldsymbol{\sigma}_k\cdot\boldsymbol{\nabla} = \partial/\partial x^k$ and the $A_k = \boldsymbol{\sigma}_k\cdot A$ are components of the vector potential $\mathbf{A}$. We use the same notation for the corresponding operators on quaternion wave functions:

$$\hat{p}_k\psi \equiv\ -\hbar\partial_k\psi i\boldsymbol{\sigma}_3 - (e/c)A_k\psi\,. \tag{A19}$$

Using the $\hat{p}_k$, components of the *Schrödinger current* $\rho\mathbf{u}$ are defined in terms of the wave function by

$$m\rho u_k \equiv (\psi^\dagger\hat{p}_k\psi)_0 = (1/2)[\Psi^\dagger\hat{p}_k\Psi + (\hat{p}_k\Psi)^\dagger\Psi]\,, \tag{A20}$$

where $m$ is the electron mass.

In the PS theory, it is appropriate to identify the *momentum operator* not as the $\hat{p}_k$ but as a related operator $\hat{\mathbf{p}}$ defined by

$$\hat{\mathbf{p}}\psi \equiv [\hat{\mathbf{p}} - (e/c)\mathbf{A}]\psi = \boldsymbol{\sigma}_k\hat{p}_k\psi\,, \tag{A21}$$

which, in the absence of magnetic interactions, reduces to

$$\hat{\mathbf{p}}\psi \equiv -\hbar\boldsymbol{\nabla}\psi i\boldsymbol{\sigma}_3\,, \tag{A22}$$

with $\boldsymbol{\nabla} = \boldsymbol{\sigma}_k\partial_k$. Using $\hat{\mathbf{p}}$, we define the *Pauli current* $\rho\mathbf{v}$ in terms of the function $\psi$ by

$$m\rho\mathbf{v} \equiv [(\hat{\mathbf{p}}\psi)\psi^\dagger]_1\,, \tag{A23}$$

where the subscript 1 calls for the vector part of the quantity in parenthesis. The action of the differential operator $\hat{\mathbf{p}}$ on the wave function $\psi$ is expressed as an algebraic operation by writing

$$\hat{\mathbf{p}}\psi = [m\mathbf{v} + imc\beta]\psi\,, \tag{A24}$$

where $\beta$ is a dimensionless scalar. Equation (A24) has the form of an eigenvalue equation with the quantity in brackets as eigenvalue. It is generally interpreted so in quantum mechanics only when $\mathbf{v}$ is constant (in which case $\beta = 0$ and $\psi$ is a "plane wave"); then the constant eigenvalue $m\mathbf{v}$ is interpreted as an "observable," namely, the momentum of

the electron. This is not inconsistent with our view that $m\rho\mathbf{v}$ is properly interpreted as a *momentum density* for any wave function $\psi$.

The momentum operator $\hat{\mathbf{p}}$ is related to the Schrödinger current by

$$\hat{p}_k\psi = [mu_k - i\rho^{-1}\partial_k(\rho\mathbf{s})]\psi\,, \tag{A25}$$

$$\hat{\mathbf{p}}\psi = [m\mathbf{u} - i\rho^{-1}\boldsymbol{\nabla}(\rho\mathbf{s})]\psi\,, \tag{A26}$$

where $\rho\mathbf{s}$ is the spin density defined by (A2). Comparing (A24) and (A26), we find that the Schrödinger and Pauli currents are related by

$$m\rho\mathbf{v} = m\rho\mathbf{u} + \boldsymbol{\nabla}\times(\rho\mathbf{s})\,, \tag{A27}$$

and that $\beta$ is given by

$$\beta = -(mc\rho)^{-1}\boldsymbol{\nabla}\cdot(\rho\mathbf{s})\,. \tag{A28}$$

The components of the conventional angular momentum operator $\hat{L}_k$ are determined by the $\hat{p}_k$ through the relation

$$\hat{L}_k = \boldsymbol{\sigma}_k\cdot(\mathbf{r}\times\boldsymbol{\sigma}_j)\hat{p}_j\,. \tag{A29}$$

From (A20) we find

$$\boldsymbol{\sigma}_k\cdot\mathbf{r}\times(m\rho\mathbf{u}) = (\psi^\dagger\hat{L}_k\psi)_0 = (1/2)[\Psi^\dagger\hat{L}_k\Psi + (\hat{L}_k\Psi)^\dagger\Psi]\,. \tag{A30}$$

Thus, the angular momentum density determined by the $\hat{L}_k$ is exactly the moment of the Schrödinger current. Precisely how this differs from the moment of the Pauli current can be determined by taking moments of (A27), with the result

$$\rho\mathbf{r}\times(m\mathbf{v}) = \rho\mathbf{r}\times(m\mathbf{u}) + 2\rho\mathbf{s} + \partial_k\{\rho\mathbf{r}\times(\boldsymbol{\sigma}_k\times\mathbf{s})\}\,. \tag{A31}$$

It is neither convenient nor necessary to associate the angular momentum density $\rho\mathbf{r}\times(m\mathbf{v})$ with a differential operator.

From the momentum operator $\hat{\mathbf{p}}$ we obtain the kinetic energy operator $(2m)^{-1}\hat{\mathbf{p}}^2$. Using (A19), (A25), and (A2), this operator can be expanded as follows:

$$\begin{aligned}\frac{1}{2m}\hat{\mathbf{p}}^2\psi &= \frac{1}{2m}\hat{p}^2\psi - \frac{e}{mc}\mathbf{B}\mathbf{s}\psi \\ &= \frac{1}{2m}\hat{p}^2\psi - \frac{e}{mc}\Big[m\mathbf{A}\cdot\mathbf{u} - i\rho^{-1}\mathbf{A}\cdot\boldsymbol{\nabla}(\rho\mathbf{s}) + \frac{e}{2c}\mathbf{A}^2 + \mathbf{B}\mathbf{s}\Big]\psi\,,\end{aligned} \tag{A32}$$

where $\hat{p}^2 \equiv \hat{p}_k\hat{p}_k$. From (A32) we get the following expressions for the *kinetic-energy density:*

$$\left(\psi^\dagger\frac{\hat{\mathbf{p}}^2\psi}{2m}\right)_0 = \left(\psi^\dagger\frac{\hat{p}^2\psi}{2m}\right)_0 - \rho\frac{e}{mc}\mathbf{B}\cdot\mathbf{s}\,, \tag{A33}$$

$$\begin{aligned}\left(\psi^\dagger\frac{\hat{p}^2\psi}{2m}\right)_0 &= \left(\psi^\dagger\frac{\hat{p}^2}{2m}\psi\right)_0 - \rho\frac{e}{mc}(m\mathbf{u}\cdot\mathbf{A} + \mathbf{s}\cdot\mathbf{B}) - \rho\frac{e^2}{2mc^2}\mathbf{A}^2 \\ &= \left(\psi^\dagger\frac{\hat{p}^2}{2m}\psi\right)_0 - \rho\frac{e}{c}\mathbf{A}\cdot\mathbf{v} + \partial_k\Big[\frac{e}{mc}\mathbf{A}\cdot(\boldsymbol{\sigma}_k\times\rho\mathbf{s})\Big] - \frac{\rho e^2}{2mc^2}\mathbf{A}^2\,.\end{aligned} \tag{A34}$$

We are especially interested in an expression for the energy density in terms of the Pauli current. From (A24) we find

$$(\hat{\mathbf{p}}\psi)^\dagger = m\psi^\dagger[\mathbf{v} - imc\beta]\,.$$

Hence,

$$\rho m^2[\mathbf{v}^2 + c^2\beta^2] = (\psi^\dagger\hat{\mathbf{p}}^\dagger)\hat{\mathbf{p}}\psi = (\psi^\dagger\hat{\mathbf{p}}^2\psi)_0 + [\boldsymbol{\nabla}(\hat{\mathbf{p}}\psi(\hbar i\boldsymbol{\sigma}_3)\psi^\dagger)]_0\,.$$

Expressing the last term in terms of observables, we get

$$[\psi^\dagger\hat{\mathbf{p}}^2\psi]_0 = (1/2)m\rho(\mathbf{v}^2 + c^2\beta^2) + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}\times\mathbf{s} + \rho c\beta\mathbf{s})\,. \tag{A35}$$

## EQUATIONS OF MOTION

In terms of the momentum operator $\hat{\mathbf{p}}$, the PS equation for the quaternion wave function is

$$\partial_t\psi i\boldsymbol{\sigma}_3\hbar = [(2m)^{-1}\hat{\mathbf{p}}^2 + e\phi]\psi\,. \tag{A36}$$

The *energy density* $\rho E$ is identified as the quantity

$$\rho E \equiv \hbar(\partial_t\psi i\boldsymbol{\sigma}_3\psi^\dagger)_0 = (1/2)i'\hbar[\Psi^\dagger\partial_t\Psi - \partial_t\Psi^\dagger\Psi]\,. \tag{A37}$$

From the PS equation (A36), then,

$$\rho E = (2m)^{-1}(\psi^\dagger\hat{\mathbf{p}}^2\psi)_0 + \rho e\phi\,. \tag{A38}$$

With $\rho e\phi$ identified as *potential-energy density*, Eq. (A38) justifies the identification of the kinetic-energy operator and density made earlier.

The energy equation (A38) is only part of the information in the PS equation. In Ref. 2 this information was expressed as the following set of equations for the Schrödinger current and the spin:

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0\,, \tag{A39a}$$

$$m\rho(\partial_t + \mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \rho e[\mathbf{E} + c^{-1}\mathbf{u}\times\mathbf{B} + (mc)^{-1}\boldsymbol{\sigma}_k\mathbf{s}\cdot\partial_k\mathbf{B}] + \partial_k\mathbf{T}_k\,, \tag{A39b}$$

$$\rho(\partial_t + \mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{s} = (\rho e/mc)\mathbf{s}\times\mathbf{B} + \partial_k\mathbf{M}_k\,, \tag{A39c}$$

where the *momentum flux* $\mathbf{T}_k$ and the *spin flux* $\mathbf{M}_k$ are the following functions of spin and density alone:

$$\mathbf{T}_k = m^{-1}\rho\boldsymbol{\sigma}_j\mathbf{s}\cdot\partial_j[\rho^{-1}\partial_k(\rho\mathbf{s})]\,, \tag{A40a}$$

$$\mathbf{M}_k = m^{-1}\rho\mathbf{s}\times\partial_k\mathbf{s}\,. \tag{A40b}$$

There is no known interpretation for the specific form of these flux terms.

Using (A27) the Schrödinger current $\rho\mathbf{u}$ can be replaced by the Pauli current $\rho\mathbf{v}$ in (A39a)–(A39c), with the result

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\,, \tag{A41a}$$

$$m\rho(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \rho e[\mathbf{E} + c^{-1}\mathbf{v}\times\mathbf{B}] + \partial_k\mathbf{T}'_k\,, \tag{A41b}$$

$$\rho(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{s} = (\rho e/mc)\mathbf{s}\times\mathbf{B} + \partial_k \mathbf{M}'_k\,, \tag{A41c}$$

where the new flux terms are related to the old ones by

$$\mathbf{T}'_k = \mathbf{T}_k + \frac{e}{mc}\mathbf{B}\times(\boldsymbol{\sigma}_k\times\rho\mathbf{s}) + \boldsymbol{\sigma}_k\times\partial_t(\rho\mathbf{s}) + m\rho[\mathbf{v}\mathbf{v}\cdot\boldsymbol{\sigma}_k - \mathbf{u}\mathbf{u}\cdot\boldsymbol{\sigma}_k]\,, \tag{A42a}$$

$$\mathbf{M}'_k = \mathbf{M}_k + m^{-1}\mathbf{s}\cdot(\boldsymbol{\nabla}\times\rho\mathbf{s})\,. \tag{A42b}$$

The last term in (A42a) comes from changing the convective derivative $(\partial_t + \mathbf{u}\cdot\boldsymbol{\nabla})$ to $(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})$; the vector $\mathbf{u}$ in that term can be eliminated using (A27), but the resulting complicated expression is unenlightening.

Equations (A41a)–(A41c) can be interpreted as conservation laws for mass and charge, momentum, and spin respectively. In principle, this set of coupled nonlinear equations for observables could be solved directly to determine the electron motion. But, of course, it is much easier to solve the linear PS equation.

Since Eqs. (A41a)–(A41c) are mathematically equivalent to Eqs. (A39a)–(A39c), the difference between the two sets of equations is entirely a matter of physical interpretation. The most significant difference is between the "body force" terms of Eqs. (A39b) and (A41b). The explicit dependence of the body force on spin in (A39b) has been eliminated in (A41b) by using the identity

$$\rho[m\mathbf{u}\times\mathbf{B} + \boldsymbol{\sigma}_k(\partial_k\mathbf{B})\cdot\mathbf{s}] = \rho m\mathbf{v}\times\mathbf{B} + \partial_k[\mathbf{B}\times(\boldsymbol{\sigma}_k\times\rho\mathbf{s})]\,, \tag{A43}$$

derived from (A27). According to (A41b), the interaction of the magnetic field with spin affects the velocity $\mathbf{v}$ of the electron only indirectly by altering the momentum flux. If the body force is interpreted as the direct interaction of the electron with external electromagnetic fields, Eq. (A41b) says that it is exactly the classical Lorentz force.

## EXPECTATION VALUES

We define the *expectation* (or *average*) *value* of an operator $\widehat{Q}$ by

$$\langle\,\widehat{Q}\,\rangle \equiv \int dV(\widehat{Q}\psi)\psi^\dagger\,, \tag{A44}$$

where the integral is over the entire region in which $\psi \neq 0$ with $\psi = 0$ on the boundary. If $\widehat{Q}$ is not a differential operator, but merely a multivector-valued function $Q = Q(\mathbf{x}, t)$, then (A45) reduces to the conventional definition

$$\langle\,Q\,\rangle \equiv \int dV Q\rho\,. \tag{A45}$$

The wave function is of course normalized so that $\langle\,1\,\rangle = 1$. The definition (A44) is equivalent to the conventional definition of real expectation values for hermitian operators in quantum mechanics. Thus, for the components of the operator $\hat{p}_k$, we have, from Eq. (A20),

$$\langle\,\hat{p}_k\,\rangle_0 = \int dV(\psi^\dagger p_k\psi)_0 = \int dV\,\Re(\Psi^\dagger\hat{p}_k\Psi)_0 = m\langle\,\mathbf{u}\,\rangle\cdot\sigma_k\,. \tag{A46}$$

It is only the vector expectation of the momentum operator

$$\langle\, \hat{\mathbf{p}} \,\rangle = \int dV[(\hat{\mathbf{p}}\psi)\psi^\dagger]_1 \,. \tag{A47}$$

Then, from (A23) and (A27), we get

$$\langle\, \hat{\mathbf{p}} \,\rangle = m\langle\, \mathbf{v} \,\rangle = m\langle\, \mathbf{u} \,\rangle = m\langle\, \hat{p}_k \,\rangle_0 \boldsymbol{\sigma}_k \,. \tag{A48}$$

Similarly, we are concerned only with the scalar expectation values of the kinetic-energy operator, so we write

$$\langle\, \hat{\mathbf{p}}^2 \,\rangle = \int dV[(\hat{\mathbf{p}}^2\psi)\psi^\dagger]_0 = \int dV(\psi^\dagger \hat{\mathbf{p}}^2 \psi)_0 \,. \tag{A49}$$

## REFERENCES


[1] D. Blokhintsev, *Principles of Quantum Mechanics* (Allyn and Bacon, Boston, 1964) discusses the Pauli theory in more detail than most texts.

[2] D. Hestenes and R. Gurtler, *Am. J. Phys.* **39**, 1028 (1971).

[3] R. Gurtler and D. Hestenes, *J. Math. Phys.* **16**, 573 (1975).

[4] D. Hestenes, *Am. J. Phys.* **39**, 1013 (1971).

[5] For further discussion of this point see C. Slichter, *Principles of Magnetic Resonance*, (Harper and Row, New York, 1963), p. 15.

[6] J. Synge, *Found. Phys.* **2**, 35 (1972).

[7] A. Messiah, *Quantum Mechanics* (Wiley, New York, 1961), Vol. **1**. p. 446.

[8] L. Ballentine, *Rev. Mod. Phys.* **42**, 358 (1970); *Am. J. Phys.* **40**, 1763 (1972).

[9] E. Wigner, *Phys. Rev.* **40**, 729 (1930): J. Moyal, *Proc. Camb. Philos. Soc.* **45**, 99 (1949).

[10] D. Hestenes, *J. Math. Phys.* **8**, 798 (1967); **16**, 568 (1975).

[11] G. Casanova, *Comptes Rendus Acad. Sci.* (Paris) **270**, A1201 (1970); **270**, A1470 (1970).

[12] W. Thirring, *Principles of Quantum Electrodynamics* (Academic, New York, 1958), p. 4.

[13] For some acute analysis, see M. Bunge, *Am. J. Phys.* **24**, 272 (1956); *Brit. J. Philos. Sci.* **6**, 1 (1955); **6**, 141 (1955).

[14] A. Lande, *Am. J. Phys.* **33**, 123 (1965); **43**, 701 (1975); *New Foundations of Quantum Mechanics* (Cambridge University, Cambridge, 1965).

[15] W. Duane, *Proc. Nat. Acad. Sci.* **9**, 158 (1923); A. Compton. *Proc. Nat. Acad. Sci.* **11**, 359 (1923).

[16]For example, see J. Krumhansl, *Perspectives in Modern Physics* (Wiley Interscience, New York, 1966), p. 534.

[17]F. London, *Z. Phys.* **63**, 245 (1930).

[18]H. Margenau and N. Kestner, *Theory of Intermolecular Forces* (Pergamon, New York, 1969), p. 22.

[19]V. Weiskopf, *Phys. Rev.* **56**, 72 (1939).

[20]M. Scully and M. Sargent III, *Phys. Today* **25**, 38 (March 1972).

[21]H. Heitler, *The Quantum Theory of Radiation*, 3rd ed. (Oxford University, London, 1950), p. 76.

[22]J. Slater, *Nature* **113**, 307 (1924).

[23]The influence of Slater's idea can easily be traced through the important papers collected together with a helpful commentary by B. L. Van der Waerden, in *Sources of Quantum Mechanics* (North-Holland, Amsterdam, 1967).

[24]H. Casimir, *Proc. Nederel. Akad. Wetensch.* **51**, 793 (1948). For recent discussions based on ideas which have much in common with those expressed here, see T. Marshall, *Nuovo Cimento* **38**, 206 (1965); L. Henry and T. Marshall, *ibid.* **41**, 188 (1966); T. Boyer, *Phys. Rev.* **174**, 1631 (1968); *Ann. Phys.* (NY) **56**, 474 (1970).

[25]M. Sparnaay, *Physica* **24**, 751 (1958).

[26]Compare with H. Casimir and D. Polder, *Phys. Rev.* **73**, 360 (1948); E. Lifschitz, *Sov. Phys.* JETP **2**, 73 (1956).

[27]D. Hestenes, *Am. J. Phys.* **38**, 840 (1970).

[28]The original derivation was given in Ref. 10, but the treatment in Ref. 3 [following the analysis in *J. Math. Phys.* **16**, 556 (1975)] is probably easier to relate to the conventional formulation of quantum mechanics.

[29]D. Hestenes, *J. Math. Phys.* **8**, 809 (1967).

[30]D. Hestenes, *J. Math. Phys.* **14**, 893 (1973).

[31]H. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden-Day, San Francisco, 1968).