Spacetime Physics with Geometric Algebra[1]

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This is an introduction to *spacetime algebra* (STA) as a unified mathematical language for physics. STA simplifies, extends and integrates the mathematical methods of classical, relativistic and quantum physics while elucidating geometric structure of the theory. For example, STA provides a single, matrix-free spinor method for rotational dynamics with applications from classical rigid body mechanics to relativistic quantum theory – thus significantly reducing the mathematical and conceptual barriers between classical and quantum mechanics. The entire physics curriculum can be unified and simplified by adopting STA as the standard mathematical language. This would enable early infusion of spacetime physics and give it the prominent place it deserves in the curriculum.

I. Introduction

Einstein's Special Theory of Relativity has been incorporated into the foundations of theoretical physics for the better part of a century, yet it is still treated as an add-on in the physics curriculum. Even today, a student can get a PhD in physics with only a superficial knowledge of Relativity Theory and its import. I submit that this sorry state of affairs is due, in large part, to serious language barriers. The standard tensor algebra of relativity theory so differs from ordinary vector algebra that it amounts to a new language for students to learn. Moreover, it is not adequate for relativistic quantum theory, which introduces a whole new language to deal with spin and quantization. The learning curve for this language is so steep that only graduate students in theoretical physics ordinarily attempt it. Thus, most physicists are effectively barred from a working knowledge of what is purported to be the most fundamental part of physics. Little wonder that the majority is content with the nonrelativistic domain for their research and teaching.

Beyond the daunting language barrier, tensor algebra has certain practical limitations as a conceptual tool. Aside from its inability to deal with spinors, standard tensor algebra is coordinate-based in an essential way, so much time must be devoted to proving covariance of physical quantities and equations. This reinforces reliance on coordinates in the physics curriculum, and it obscures the fundamental role of geometric invariants in physics. We can do better – much better!

This is the second in a series of articles introducing *geometric algebra* (GA) as a unified mathematical language for physics. The first article[1] (hereafter

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referred to as GA1) shows how GA simplifies and unifies the mathematical methods of classical physics and nonrelativistic quantum mechanics. This article extends that unification to spacetime physics by developing a *spacetime algebra* (STA) expressly designed for that purpose. A third article is planned to present a profound and surprising extension of the language to incorporate General Relativity.[2]

Although this article provides a self-contained introduction to STA, the serious reader is advised to study GA1 first for background and motivation. This is not a primer on relativity and quantum mechanics. Readers are expected to be familiar with those subjects so they can make their own comparisons of standard approaches to the topics treated here. Topics have been selected to showcase unique advantages of STA rather than for balanced coverage of every subject. Nevertheless, topics are developed in sufficient detail to make STA useful in instruction and research, at least after some practice and consultation with the literature. The general objectives of each Section in the article can be summarized as follows:

Section II presents the defining grammar for STA and introduces basic definitions and theorems needed for coordinate-free formulation and application of spacetime geometry to physics.

Section III distinguishes between *proper* (invariant) and *relative* formulations of physics. It introduces a simple algebraic device called the *spacetime split* to relate proper descriptions of physical properties to relative descriptions with respect to inertial systems. This provides a seamless connection of STA to the GA of classical physics in GA1.

Section IV extends the treatment of rotations and reflections in GA1 to a coordinate-free treatment of Lorentz transformations on spacetime. The method is more versatile than standard methods, because it applies to spinors as well as vectors, and it reduces the composition of Lorentz transformations to the geometric product.

Lorentz *invariant* physics with STA obviates any need for the passive Lorentz transformations between coordinate systems that are required by standard *covariant* formulations. Instead, Section V uses the spinor form of an active Lorentz transformation to characterize change of state along world lines. This generalizes the spinor treatment of classical rigid body mechanics in GA1, so it articulates smoothly with nonrelativistic theory. It has the dual advantages of simplifying solutions of the classical Lorentz force equation while generalizing it to a classical model of an electron with spin that is shown to be a classical limit of the Dirac equation in Section VIII.

Section VI shows how STA simplifies electromagnetic field theory, including reduction of Maxwell's equations to a single invertible field equation. It is most notable that this simplification comes from recognizing that the famous "Dirac operator" is just the STA derivative with respect to a spacetime point, so it is as significant for Maxwell's equation as for Dirac's equation.

Section VII reformulates Dirac's famous equation for the electron in terms of the real STA, thereby showing that complex numbers are superfluous in relativistic quantum theory. STA reveals geometric structure in the Dirac wave

function that has long gone unrecognized in the standard matrix theory. That structure is explicated and analyzed at length to ascertain implications for the interpretation of quantum theory.

Section VIII discusses alternatives to the Copenhagen interpretation of quantum mechanics that are motivated by geometric analysis of the Dirac theory. The questions raised by this analysis may be more important than the conclusions. My own view is that the Copenhagen interpretation cannot account for the structure of the Dirac theory, but a fully satisfactory alternative remains to be found.

Finally, Section IX outlines how STA can streamline the physics curriculum to give the powerful ideas of relativistic field theory and quantum mechanics roles that are commensurate with their importance.

II. Spacetime Algebra

The standard model for spacetime is a real 4D *Minkowski vector space* $\mathcal{M}^4$ called *Minkowski spacetime* or (by suppressing the distinction between the model and the physical reality it is supposed to represent) simply *spacetime*. With vector addition and scalar multiplication taken for granted, we impose the geometry of spacetime on $\mathcal{M}^4$ by defining the *geometric product* uv for vectors u, v, w by the following rules:

$$(uv)w = u(vw)\,, \qquad \text{associative} \tag{1}$$

$$u(v+w) = uv + uw\,, \qquad \text{left distributive} \tag{2}$$

$$(v+w)u = vu + wu\,, \qquad \text{right distributive} \tag{3}$$

$$v^2 = \epsilon_v |\, v \,|^2\,, \qquad \text{contraction} \tag{4}$$

where ϵ_v is the *signature* of v and the *magnitude* $|\, v \,|$ is a real positive scalar. As usual in spacetime physics, we say that v is *timelike* if its signature is positive ($\epsilon_v = 1$), *spacelike* if ($\epsilon_v = -1$), or *lightlike* if $|\, v \,| = 0$, which is equivalent to null signature ($\epsilon_v = 0$).

It should be noted that these are the same rules defining the "classical geometric algebra" in GA1, except for the signature in the contraction rule (4) that allows vectors to have negative or null square. (This modification was the great innovation of Minkowski that we honor by invoking his name!)

Spacetime vectors are denoted by italic letters to distinguish them from the 3D vectors denoted by boldface letters in GA1. This convention is especially helpful when we formulate relations between the two kinds of vector in Section III.

By successive multiplications and additions, the vectors of $\mathcal{M}^4$ generate a geometric algebra $\mathcal{G}_4 = \mathcal{G}(\mathcal{M}^4)$ called *spacetime algebra* (STA). As usual in a geometric algebra, the elements of $\mathcal{G}_4$ are called *multivectors*. The above rules defining the geometric product are the basic grammar rules of STA.

In reviewing its manifold applications to physics, one can see that STA derives astounding power and versatility from

- the *simplicity* of its grammar,
- the *geometric meaning of multiplication*,
- the way *geometry links the algebra to the physical world*.

As we have seen before, the geometric product uv can be decomposed into a symmetric *inner product*

$$u \cdot v = \tfrac{1}{2}(uv + vu) = v \cdot u, \tag{5}$$

and an antisymmetric *outer product*

$$u \wedge v = \tfrac{1}{2}(uv - vu) = -v \wedge u. \tag{6}$$

so that

$$uv = u \cdot v + u \wedge v\,. \tag{7}$$

To facilitate coordinate-free manipulations in STA, it is useful to generalize the inner and outer products of vectors to arbitrary multivectors. We define the outer product along with the notion of *k-vector* iteratively as follows: Scalars are defined to be 0-vectors, vectors are 1-vectors, and bivectors, such as $u \wedge v$, are 2-vectors. For a given k-vector K, the integer k is called the *step* (or grade) of K. For $k \geq 1$, the outer product of a vector v with a k-vector K is a $(k+1)$-vector defined in terms of the geometric product by

$$v \wedge K = \tfrac{1}{2}(vK + (-1)^k Kv) = (-1)^k K \wedge v\,. \tag{8}$$

The corresponding inner product is defined by

$$v \cdot K = \tfrac{1}{2}(vK + (-1)^{k+1} Kv) = (-1)^{k+1} K \cdot v\,, \tag{9}$$

and it can be proved that the result is a $(k-1)$-vector. Adding (8) and (9) we obtain

$$vK = v \cdot K + v \wedge K\,, \tag{10}$$

which obviously generalizes (7). The important thing about (10), is that it decomposes vK into $(k-1)$-vector and $(k+1)$-vector parts.

A basis for STA can be generated by a *standard frame* $\{\gamma_\mu;\ 0, 1, 2, 3\}$ of orthonormal vectors, with timelike vector γ_0 in the forward light cone and components $g_{\mu\nu}$ of the usual metric tensor given by

$$g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu = \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)\,. \tag{11}$$

(We use $c = 1$ so spacelike and timelike intervals are measured in the same unit.) The γ_μ determine a unique *righthanded unit pseudoscalar*

$$i = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3\,. \tag{12}$$

It follows that

$$i^2 = -1\,, \quad \text{and} \quad \gamma_\mu i = -i\gamma_\mu. \tag{13}$$

Thus, i is a geometrical $\sqrt{-1}$, but it anticommutes with all spacetime vectors. By forming all distinct products of the γ_μ we obtain a complete basis for the STA $\mathcal{G}_4$ consisting of the $2^4 = 16$ linearly independent elements

$$1, \quad \gamma_\mu, \quad \gamma_\mu \wedge \gamma_\nu, \quad \gamma_\mu i, \quad i\,. \tag{14}$$

To facilitate algebraic manipulations it is convenient to introduce the *reciprocal frame* $\{\gamma^\mu\}$ defined by the equations

$$\gamma_\mu = g_{\mu\nu}\gamma^\nu \qquad \text{or} \qquad \gamma_\mu \cdot \gamma^\nu = \delta_\mu^\nu\,. \tag{15}$$

(summation convention in force!) Now, any multivector can be expressed as a linear combination of the basis elements (14). For example, a bivector F has the expansion

$$F = \tfrac{1}{2}F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu\,, \tag{16}$$

with its "scalar components" $F^{\mu\nu}$ given by

$$F^{\mu\nu} = \gamma^\mu \cdot F \cdot \gamma^\nu = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F\,. \tag{17}$$

Note that the two inner products in the second form can be performed in either order, so a parenthesis is not needed.

The entire spacetime algebra is obtained by taking linear combinations of basis k-vectors in (14).

A generic element M of the STA, called a *multivector*, can therefore be written in the *expanded form*

$$M = \alpha + a + F + bi + \beta i\,, \tag{18}$$

where α and β are scalars, a and b are vectors, and F is a bivector. This is a decomposition of M into its k-vector parts, with $k = 0, 1, 2, 3, 4$, as is expressed more explicitly by putting (18) in the form

$$M = \sum_{k=0}^{4} M_{(k)}, \tag{19}$$

where the subscript (k) means "*k-vector part*." Of course, $M_{(0)} = \alpha$, $M_{(1)} = a$, $M_{(2)} = F$, $M_{(3)} = bi$, $M_{(4)} = \beta i$. Alternative notations include $M_S = \langle M \rangle = M_{(0)}$ for the scalar part of a multivector. The scalar part of a product behaves much like the "trace" in matrix algebra. For example, we have the very useful theorem $\langle MN \rangle = \langle NM \rangle$ for arbitrary M and N.

Computations are also facilitated by the operation of *reversion*, the name indicating reversal in the order of geometric products. For M in the expanded form (18) the *reverse* $\widetilde{M}$ can be defined by

$$\widetilde{M} = \alpha + a - F - bi + \beta i\,. \tag{20}$$

Note, in particular, the effect of reversion on the various k-vector parts.

$$\widetilde{\alpha} = \alpha, \quad \widetilde{a} = a, \quad \widetilde{F} = -F, \quad \widetilde{i} = i\,. \tag{21}$$

It is not difficult to prove that

$$(MN)^{\sim} = \widetilde{N}\widetilde{M}\,, \tag{22}$$

for arbitrary M and N. For example, in (20) we have $(bi)^{\sim} = ib = -bi$, where the last sign follows from (13).

A positive definite magnitude $|\,M\,|$ for any multivector M can now be defined by

$$|\,M\,|^2 = |\,\langle M\widetilde{M}\rangle\,|\,. \tag{23}$$

Any multivector M can be decomposed into the sum of an *even* part M_+ and an *odd* part M_- defined in terms of the expanded form (18) by

$$M_+ = \alpha + F + \beta i\,, \tag{24}$$

$$M_- = a + bi\,, \tag{25}$$

or, equivalently, by

$$M_\pm = \tfrac{1}{2}(M \mp iMi)\,. \tag{26}$$

The set $\{M_+\}$ of all even multivectors forms an important subalgebra of STA called the *even subalgebra*.

If ψ is an even multivector, then $\psi\widetilde{\psi}$ is also even, but its bivector part must vanish according to (20), since $(\psi\widetilde{\psi})^{\sim} = \psi\widetilde{\psi}$. Therefore, $\psi\widetilde{\psi}$ has only scalar and pseudoscalar parts, as expressed by writing

$$\psi\widetilde{\psi} = \rho e^{i\beta} = \rho(\cos\beta + i\sin\beta)\,, \tag{27}$$

where $\rho \geq 0$ and β are scalars. If $\rho \neq 0$ we can derive from ψ an even multivector $R = \psi(\psi\widetilde{\psi})^{-\frac{1}{2}}$ satisfying

$$R\widetilde{R} = \widetilde{R}R = 1\,. \tag{28}$$

Then ψ can be put in the *canonical form*

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}}R \tag{29}$$

We shall see that this *invariant decomposition* has a fundamental physical significance in the Dirac Theory.

An important special case of the decomposition (29) is its application to a bivector F, for which it is convenient to replace $\beta/2$ by $\beta + \pi/2$ and write $f = \rho^{\frac{1}{2}}Ri$. Thus, for any bivector F that is not null ($F^2 \neq 0$) we have the invariant *canonical form*

$$F = fe^{i\beta} = f(\cos\beta + i\sin\beta)\,, \tag{30}$$

where $f^2 = -f\widetilde{f} = |\,f\,|^2$, so f is said to be a *timelike bivector* with magnitude $|\,f\,|$. Similarly, the dual if is said to be a *spacelike bivector,* since $(if)^2 = -|\,f\,|^2$.

Thus the right side of (30) is the unique decomposition of F into a sum of mutually commuting timelike and spacelike parts.

When $F^2 = 0$, F is said to be a *lightlike bivector*, and it can still be written in the form (30) with

$$f = k \wedge e = ke\,, \tag{31}$$

where k is a null vector and e is a spacelike vector orthogonal to k. In this case, the decomposition is not unique, and the exponential factor can always be absorbed in the definition of f.

To extend spacetime algebra into a complete *spacetime calculus*, suitable definitions for derivatives and integrals are required. Though that can be done in a completely coordinate-free way,[6] it is more expedient here to exploit one's prior knowledge about coordinates.

For each spacetime point x a standard frame $\{\gamma_\mu\}$ determines a set of "rectangular coordinates" $\{x^\mu\}$ given by

$$x^\mu = \gamma^\mu \cdot x \qquad \text{and} \qquad x = x^\mu \gamma_\mu\,. \tag{32}$$

In terms of these coordinates the derivative with respect to a spacetime point x is an operator $\nabla \equiv \partial_x$ that can be defined by

$$\nabla = \gamma^\mu \partial_\mu, \tag{33}$$

where ∂_μ is given by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \gamma_\mu \cdot \nabla\,. \tag{34}$$

The square of ∇ is the usual d'Alembertian

$$\nabla^2 = g^{\mu\nu}\partial_\mu\partial_\nu \quad \text{where} \quad g^{\mu\nu} = \gamma^\mu \cdot \gamma^\nu. \tag{35}$$

The matrix representation of the *vector derivative* ∇ can be recognized as the socalled "Dirac operator," originally discovered by Dirac by seeking a "square root" of the d'Alembertian (35) in order to find a first order "relativistically invariant" wave equation for the electron. In STA however, where the γ^μ are vectors rather than matrices, it is clear that ∇ is a vector operator; indeed, it provides an appropriate definition for the derivative with respect to any spacetime vector variable.

Contrary to the impression given by conventional accounts of relativistic quantum theory, the operator ∇ is not specially adapted to spin-$\frac{1}{2}$ wave equations. It is equally apt for electromagnetic field equations, as seen in Section VI.

This is a good point to describe the relation of STA to the standard Dirac algebra. The *Dirac matrices* are representations of the vectors γ_μ in STA by 4×4 matrices, and to emphasize this correspondence the vectors here are denoted with the same symbols γ_μ ordinarily used to represent the Dirac matrices. In view of what we know about STA, this correspondence reveals the physical significance of the Dirac matrices, appearing so mysteriously in relativistic

quantum mechanics: *The Dirac matrices are no more and no less than matrix representations of an orthonormal frame of spacetime vectors* and thereby they characterize spacetime geometry. But how can this be? Dirac never said any such thing! And physicists today regard the set $\{\gamma_\mu\}$ as a single vector with matrices for components. Nevertheless, their practice shows that the "frame interpretation" is the correct one, though we shall see later that the "component interpretation" is actually equivalent to it in certain circumstances. The correct interpretation was actually inherent in Dirac's argument to derive the matrices in the first place: First he put the γ_μ in one-to-one correspondence with orthogonal directions in spacetime by indexing them. Second, he related the γ_μ to the metric tensor by imposing the "peculiar condition" (11) on the matrices for formal algebraic reasons. But we see in (11) that this condition has a clear geometric meaning in STA as the inner product of vectors in the frame. Finally, Dirac introduced associativity automatically by employing matrix algebra, without realizing that it has a geometric meaning in this context. If indeed the physical significance of the Dirac matrices derives entirely from their interpretation as a frame of vectors, then their specific matrix properties must be irrelevant to physics. That is proved in Section VII by dispensing with matrices altogether and formulating the Dirac theory entirely in terms of STA.

In relativistic quantum mechanics one often encounters the notation $\gamma \cdot p = \gamma^\mu p_\mu$, where γ is regarded formally as a vector with matrices γ^μ as components and p is an ordinary vector. Likewise, the Dirac operator is denoted by $\gamma \cdot \partial = \gamma^\mu \partial_\mu$ without recognizing it as a generic vector derivative with components ∂_μ. The notation $\gamma \cdot p$ has the same deficiencies as the notation $\boldsymbol{\sigma} \cdot \mathbf{a}$ criticized in GA1. In STA it is inconsistent with identification of $\{\gamma^\mu\}$ as an orthonormal frame.

III. Proper Physics and Spacetime Splits

STA makes it possible to formulate and analyze conventional relativistic physics in invariant form without reference to a coordinate system. To emphasize the distinctive features of this formulation, I like to call it "*proper physics*." From the proper point of view, the term "relativistic mechanics" is a misnomer, because the theory is less rather than more relativistic than the so-called "nonrelativistic" mechanics of Newton. The equations describing a particle in Newtonian mechanics depend on the motion of the particle *relative* to some observer; in Einstein's mechanics they do not. Einstein originally formulated his mechanics in terms of "relative variables" (such as the position and velocity of a particle relative to a given observer), but he eliminated dependence of the equations on the observer's motion by the "relativity postulate," which requires that the form of the equations be invariant under a change of relative variables from those of one inertial observer to those of another. Despite the taint of misnomer, the terms "relativistic" and "nonrelativistic" are so ensconced in the literature that it is awkward to avoid them.

Minkowski's covariant formulation of Einstein's theory replaced the explicit

use of variables relative to inertial observers by components relative to an arbitrary coordinate system for spacetime. The "*proper formulation*" given here takes another step to move from covariance to invariance by relating particle motion directly to Minkowski's "absolute spacetime" without reference to any coordinate system. Minkowski had the great idea of interpreting Einstein's theory of relativity as a prescription for fusing space and time into a single entity "spacetime".[5] The straightforward algebraic characterization of "Minkowski spacetime" by spacetime algebra makes a proper formulation of physics possible.

The *history* or *world line* of a material particle is a timelike curve $x = x(\tau)$ in spacetime. Particle conservation is expressed by assuming that the function $x(\tau)$ is single-valued and continuous except possibly at discrete points where particle creation and/or annihilation occurs. Only differentiable particle histories are considered here, and τ always refers to the *proper time* (arc length) of a particle history. After a unit of length (say centimeters) has been chosen, the physical significance of the spacetime metric is fixed by the assumption that the proper time of a material particle is equal to the time (in centimeters) recorded on a (perhaps hypothetical) clock traveling with the particle.

The unit tangent $v = v(\tau) = dx/d\tau \equiv \dot{x}$ of a particle history will be called the *(proper) velocity* of the particle. By the definition of proper time, we have $d\tau = |\, dx \,| = |\, (dx)^2 \,|^{\frac{1}{2}}$, and

$$v^2 = 1. \tag{36}$$

The term "proper velocity," is preferable to the alternative terms "world velocity," "invariant velocity," and "four velocity." The adjective "proper" is used to emphasize that the velocity v describes an intrinsic property of the particle, independent of any observer or coordinate system. The adjective "absolute" would do the same, but it may not be free from undesirable connotations. Moreover, the word "proper" is shorter and has already been used in a similar sense in the terms "proper mass" and "proper time." The adjective "invariant" is inappropriate, because no coordinates or transformation group has been introduced. The velocity should not be called a "4-vector," because that term means pseudoscalar in STA; besides, there is no need to refer to four components of the velocity.

Though STA enables us to describe physical processes by proper equations, observations and measurements are often expressed in terms of variables tied to a particular inertial system, so we need to know how to reformulate proper equations in terms of those variables. STA provides a very simple way to do that called a *spacetime split*.

In STA a given inertial system is completely characterized by a single future-pointing, timelike unit vector. Refer to the inertial system characterized by the vector γ_0 as the γ_0-*system*. The vector γ_0 is tangent to the world line of an observer at rest in the γ_0-system, so it is convenient to use γ_0 as *a name for the observer*. The observer γ_0 is represented algebraically in STA in the same way as any other physical system, and the spacetime split amounts to no more

than comparing the motion of a given system (the observer) to other physical systems. Indeed, the world line of an inertial observer is the straight world line of a free particle, so inertial frames can be characterized by free particles without the anthropomorphic reference to observers.

An *inertial observer* γ_0 determines a unique mapping of spacetime into the even subalgebra of STA. For each spacetime point (or *event*) x the mapping is specified by

$$x\gamma_0 = t + \mathbf{x}\,, \tag{37}$$

where

$$t = x \cdot \gamma_0 \tag{38}$$

and

$$\mathbf{x} = x \wedge \gamma_0\,. \tag{39}$$

This defines the γ_0-*split* of spacetime. Equation (38) assigns a unique time t to every event x; indeed, (38) is the equation for a one-parameter family of spacelike hyperplanes with normal γ_0.

Equation (39) assigns to each event x a unique *position vector* $\mathbf{x}$ in the γ_0 system. Thus, to each event x the single equation (37) assigns a unique time and position in the γ_0-system. Note that the reverse of (37) is

$$\gamma_0 x = \gamma_0 \cdot x + \gamma_0 \wedge x = t - \mathbf{x}\,, \tag{40}$$

so, since $\gamma_0^2 = 1$,

$$x^2 = (x\gamma_0)(\gamma_0 x) = (t - \mathbf{x})(t + \mathbf{x}) = t^2 - \mathbf{x}^2\,. \tag{41}$$

The form and value of this equation are independent of the chosen observer; thus we have proved that the expression $t^2 - \mathbf{x}^2$ is *Lorentz invariant* without even mentioning a Lorentz transformation. Thus, the term "Lorentz invariant" can be construed as meaning "independent of a chosen spacetime split." In contrast to (41), equation (37) is not Lorentz invariant; indeed, for a different observer γ_0' we get the split

$$x\gamma_0' = t' + \mathbf{x}'\,. \tag{42}$$

Mostly we shall work with *manifestly Lorentz invariant* equations, which are independent of even an indirect reference to an inertial system.

The set of all position vectors (39) is the 3-dimensional *position space of the observer* γ_0, which we designate by $\mathcal{P}^3 = \mathcal{P}^3(\gamma_0) = \{\mathbf{x} = x \wedge \gamma_0\}$. Note that $\mathcal{P}^3$ consists of all bivectors in STA with γ_0 as a common factor. In agreement with common parlance, we refer to the elements of $\mathcal{P}^3$ as vectors. Thus, we have two kinds of vectors, those in $\mathcal{M}^4$ and those in $\mathcal{P}^3$. To distinguish between them, we refer to elements of $\mathcal{M}^4$ as *proper vectors* and to elements of $\mathcal{P}^3$ as *relative*

vectors (relative to γ_0, of course!). To keep the discussion clear, relative vectors are designated in boldface, while proper vectors are not.

By the geometric product and sum, the vectors in $\mathcal{P}^3$ generate the entire even subalgebra of STA as the geometric algebra $\mathcal{G}_3 = \mathcal{G}(\mathcal{P}^3)$ employed for classical physics in GA1. This is made obvious by constructing a basis. Corresponding to a standard basis $\{\gamma_\mu\}$ for $\mathcal{M}^4$, we have a standard basis $\{\boldsymbol{\sigma}_k; k = 1, 2, 3\}$ for $\mathcal{P}^3$, where

$$\boldsymbol{\sigma}_k = \gamma_k \wedge \gamma_0 = \gamma_k \gamma_0 \,. \tag{43}$$

These generate a basis for the relative bivectors:

$$\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j = \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = i\boldsymbol{\sigma}_k = \gamma_j \gamma_i \,, \tag{44}$$

where the allowed values of the indices $\{i, j, k\}$ are cyclic permutations of 1,2,3, and the wedge is the outer product of relative vectors (not to be confused with the outer product of proper vectors as in (43)). The right sides of (43) and (44) show how the bivectors for spacetime are split into vectors and bivectors for $\mathcal{P}^3$. Comparison with (14) shows that the $\boldsymbol{\sigma}_k$ generate the entire even subalgebra, which can therefore be identified with $\mathcal{G}_3 = \mathcal{G}(\mathcal{P}^3)$. Remarkably, the right-handed pseudoscalar for $\mathcal{P}^3$ is identical to that for $\mathcal{M}^4$, that is,

$$\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \,. \tag{45}$$

To be consistent with the operation of *reversion* defined in GA1 for the algebra $\mathcal{G}_3$ we require

$$\boldsymbol{\sigma}_k^\dagger = \boldsymbol{\sigma}_k \qquad \text{and} \qquad (\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j)^\dagger = \boldsymbol{\sigma}_j \boldsymbol{\sigma}_i \,. \tag{46}$$

This can be extended to the entire STA by defining

$$M^\dagger \equiv \gamma_0 \widetilde{M} \gamma_0 \tag{47}$$

for an arbitrary multivector M. The explicit appearance of the timelike vector γ_0 here shows the dependence of $M^\dagger$ on a particular spacetime split. The definitions in this paragraph guarantee smooth articulation of proper physics with physical descriptions relative to inertial frames.

Now let us rapidly survey the spacetime splits of some important physical quantities. Let $x = x(\tau)$ be the history of a particle with *proper time* τ and *proper velocity* $v = dx/d\tau$. The spacetime split of v is obtained by differentiating (37); whence

$$v\gamma_0 = v_0(1 + \mathbf{v}) \,, \tag{48}$$

where

$$v_0 = v \cdot \gamma_0 = \frac{dt}{d\tau} = \left(1 - \mathbf{v}^2\right)^{-\frac{1}{2}} \tag{49}$$

is the "time dilation" factor, and

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{d\tau}{dt}\frac{d\mathbf{x}}{d\tau} = \frac{v \wedge \gamma_0}{v \cdot \gamma_0} \tag{50}$$

is the *relative velocity* in the γ_0-system. The last equality in (49) was obtained from

$$1 = v^2 = (v\gamma_0)(\gamma_0 v) = v_0(1+\mathbf{v})v_0(1-\mathbf{v}) = v_0^2(1-\mathbf{v}^2)\,. \tag{51}$$

Let p be the *proper momentum* (i.e., energy-momentum vector) of a particle. The spacetime split of p into *energy* (or relative mass) E and *relative momentum* $\mathbf{p}$ is given by

$$p\gamma_0 = E + \mathbf{p}\,, \tag{52}$$

where

$$E = p \cdot \gamma_0 \quad \text{and} \quad \mathbf{p} = p \wedge \gamma_0\,. \tag{53}$$

Of course

$$p^2 = (E+\mathbf{p})(E-\mathbf{p}) = E^2 - \mathbf{p}^2 = m^2\,, \tag{54}$$

where m is the *proper mass* of the particle.

The proper angular momentum of a particle relates its proper momentum p to its location at a spacetime point x. Performing the splits as before, we find

$$px = (E+\mathbf{p})(t-\mathbf{x}) = Et + \mathbf{p}t - E\mathbf{x} - \mathbf{p}\mathbf{x}\,. \tag{55}$$

The scalar part of this gives the familiar split

$$p \cdot x = Et - \mathbf{p} \cdot \mathbf{x}\,, \tag{56}$$

so often employed in the phase of a wave function. The bivector part gives us the proper angular momentum

$$p \wedge x = \mathbf{p}t - E\mathbf{x} + i(\mathbf{x} \times \mathbf{p})\,, \tag{57}$$

where, as explained in GA1, $\mathbf{x} \times \mathbf{p}$ is the standard vector cross product.

An electromagnetic field is a bivector-valued function $F = F(x)$ on spacetime. An observer γ_0 splits it into an electric (relative vector) part $\mathbf{E}$ and, a magnetic (relative bivector) part $i\mathbf{B}$; thus

$$F = \mathbf{E} + i\mathbf{B}\,, \tag{58}$$

where

$$\mathbf{E} = (F \cdot \gamma_0)\gamma_0 = \tfrac{1}{2}(F + F^\dagger) \tag{59}$$

is the part of F that anticommutes with γ_0, and

$$i\mathbf{B} = (F \wedge \gamma_0)\gamma_0 = \tfrac{1}{2}(F - F^\dagger) \tag{60}$$

is the part that commutes. Also, in accordance with (47), $F^\dagger = \mathbf{E} - i\mathbf{B}$. Note that the split of the electromagnetic field in (58) corresponds exactly to the split of the angular momentum (57) into relative vector and bivector parts.

A different kind of spacetime split is most appropriate for Lorentz transformations, as explained in the next Section.

IV. Lorentz Transformations

Orthogonal transformations on spacetime are called *Lorentz transformations.* With due attention to the indefinite signature of spacetime (11), *geometric algebra enables us to treat Lorentz transformations by the same coordinate-free methods used in GA1 for 3D rotations and reflections.* Again, the method has the great advantage of reducing the composition of transformations to simple versor multiplication. The method is developed here in complete generality to include space and time inversion, but the emphasis is on rotors and rotations as a foundation for classical spinor mechanics in the next Section and subsequent connection to relativistic quantum mechanics in Section VIII.

The main theorem is that any Lorentz transformation of a spacetime vector a can be expressed in the *canonical form*

$$\underline{L}a = \epsilon_L L a L^{-1}, \tag{61}$$

where $\epsilon_L = 1$ if versor L is an even multivector and $\epsilon_L = -1$ if L is odd. The condition

$$LL^{-1} = 1 \tag{62}$$

allows L to have any nonzero magnitude, but normalization to $|\, L \,| = 1$ is often convenient. The Lorentz transformation $\underline{L}$ is said to be *proper* if $\epsilon_L = 1$, and *improper* if $\epsilon_L = -1$. It is said to be *orthochronous* if, for any timelike vector v,

$$v \cdot \underline{L}(v) > 0\,. \tag{63}$$

A *proper, orthochronous* Lorentz transformation is called a *Lorentz rotation* (or a *restricted* Lorentz transformation). For a Lorentz rotation $\underline{R}$ the canonical form can be written

$$\underline{R}(a) = Ra\widetilde{R}\,, \tag{64}$$

where the even multivector R is called a *rotor* and is normalized by the condition

$$R\widetilde{R} = 1\,. \tag{65}$$

The rotors form a multiplicative group called the *rotor group*, which is a double-valued representation of the Lorentz rotation group (also called the restricted Lorentz group).

As in the 3D case, the canonical form (61) simplifies the whole treatment of Lorentz transformations. In particular, its main advantage is that it reduces the composition law for Lorentz transformations,

$$\underline{L}_2\, \underline{L}_1 = \underline{L}_3 \tag{66}$$

to the versor product

$$L_2\, L_1 = L_3. \tag{67}$$

It follows from the rotor form (64), that, for any vectors a and b,

$$(\underline{R}a)(\underline{R}b) = Rab\widetilde{R} = \underline{R}(ab). \tag{68}$$

Thus, Lorentz rotations preserve the geometric product. This implies that the Lorentz rotation (64) can be extended to any multivector M as

$$\underline{R}M = RM\widetilde{R}. \tag{69}$$

The most elementary kind of Lorentz transformation is a *reflection* $\underline{n}$ by a (non-null) vector n, according to

$$\underline{n}(a) = -nan^{-1}\,. \tag{70}$$

This is a reflection with respect to a hyperplane with normal n. Even if n is normalized to $|\,n\,| = 1$, if it is spacelike we need $n^{-1} = -n$ in (70) to account for its negative signature. A reflection

$$\underline{v}(a) = -vav \tag{71}$$

with respect to a timelike vector $v = v^{-1}$ is called a *time reflection*. Let n_1, n_2, n_3 be spacelike vectors that compose the trivector

$$n_3n_2n_1 = iv\,. \tag{72}$$

A *space inversion* $\underline{v}_s$ can then be defined as the composite of reflections with respect to these three vectors, so it can be written

$$\underline{v}_s(a) = n_3n_2n_1a\,n_1n_2n_3 = ivavi = vav\,. \tag{73}$$

Note the difference in sign between the right sides of (71) and (73). The composite of the time reflection (71) with the space inversion (73) is the *spacetime inversion*

$$\underline{v}_{st}(a) = \underline{v}_s v(a) = iai^{-1} = -a\,, \tag{74}$$

which is represented by the pseudoscalar i. Note that spacetime inversion is proper but not orthochronous, so it is not a rotation despite the fact that i is even.

Two basic types of Lorentz rotation can be obtained from the product of two reflections, namely *timelike rotations* (or *boosts*) and *spacelike rotations*. For a *boost*

$$\underline{L}(a) = La\,\widetilde{L}\,, \tag{75}$$

the rotor L can be factored into a product

$$L = v_2v_1 \tag{76}$$

of two unit timelike vectors v_1 and v_2. The boost is a rotation in the timelike plane containing v_1 and v_2. The factorization (76) is not unique. Indeed, for a given L any timelike vector in the plane can be chosen as v_1, and v_2 can then be computed from (76). Similarly, for a *spacelike rotation*

$$\underline{U}(a) = Ua\widetilde{U}, \tag{77}$$

the rotor U can be factored into a product

$$U = n_2 n_1 \tag{78}$$

of two unit spacelike vectors in the *spacelike plane* of the rotation. Note that the product, say $n_2 v_1$, of a spacelike vector with a timelike vector is not a rotor, because the corresponding Lorentz transformation is not orthochronous. Likewise, the pseudoscalar i is not a rotor, even though it can be expressed as the product of two bivectors, for it does not satisfy the rotor condition $R\widetilde{R} = 1$.

The Lorentz rotation (64) can be applied to a standard frame $\{\gamma_\mu\}$, transforming it into a new frame of vectors $\{e_\mu\}$ given by

$$e_\mu = R\gamma_\mu\widetilde{R}\,. \tag{79}$$

A *spacetime rotor split* of this Lorentz rotation is accomplished by a split of the rotor R into the product

$$R = LU\,, \tag{80}$$

where $U^\dagger = \gamma_0\widetilde{U}\gamma_0 = \widetilde{U}$ or

$$U\gamma_0\widetilde{U} = \gamma_0 \tag{81}$$

and $L^\dagger = \gamma_0\widetilde{L}\gamma_0 = L$ or

$$\gamma_0\widetilde{L} = L\gamma_0\,. \tag{82}$$

This determines a split of (79) into a sequence of two Lorentz rotations determined by U and L respectively; thus,

$$e_\mu = R\gamma_\mu\widetilde{R} = L(U\gamma_\mu\widetilde{U})\widetilde{L}\,. \tag{83}$$

In particular, by (81) and (82),

$$e_0 = R\gamma_0\widetilde{R} = L\gamma_0\widetilde{L} = L^2\gamma_0\,. \tag{84}$$

Hence,

$$L^2 = e_0\gamma_0\,. \tag{85}$$

This determines L uniquely in terms of the timelike vectors e_0 and γ_0, which, in turn, uniquely determines the split (80) of R, since U can be computed from $U = \widetilde{L}R$.

It is essential to note that the "spacetime rotor split" (80) is quite different from the "spacetime split" introduced in the preceding section, for example in (58). The terminology is motivated by the expression of rotors U and L in terms of relative vectors, to which we now turn.

Equation (81) for variable U defines the *"little group"* of Lorentz rotations that leave γ_0 invariant; This is the group of *"spatial rotations"* in the γ_0-system. Each such rotation takes a frame of proper vectors γ_k (for $k = 1, 2, 3$) into a new frame of vectors $U\gamma_k\widetilde{U}$ in the γ_0-system. Multiplication by γ_0 expresses this as a rotation of relative vectors $\boldsymbol{\sigma}_k = \gamma_k\gamma_0$ into relative vectors $\mathbf{e}_k$; thus, we get

$$\mathbf{e}_k = U\boldsymbol{\sigma}_k U^\dagger = U\boldsymbol{\sigma}_k\widetilde{U}\,, \tag{86}$$

in exact agreement with the equation for 3D rotations in GA1.

Equation (84) can be solved for L, in particular, for the case where $e_0 = v$ is the proper velocity of a particle of mass m. Then (48) enables us to write (85) in the alternative forms

$$L^2 = v\gamma_0 = \frac{p\gamma_0}{m} = \frac{E + \mathbf{p}}{m}\,, \tag{87}$$

It is easily verified that this has the solution

$$\begin{aligned} L = (v\gamma_0)^{\frac{1}{2}} &= \frac{1 + v\gamma_0}{\big[2(1 + v\cdot\gamma_0)\big]^{\frac{1}{2}}} \\ &= \frac{m + p\gamma_0}{\big[2m(m + p\cdot\gamma_0)\big]^{\frac{1}{2}}} = \frac{m + E + \mathbf{p}}{\big[2m(m + E)\big]^{\frac{1}{2}}}\,. \end{aligned} \tag{88}$$

This displays L as a boost of a particle from rest in the γ_0-system to a relative momentum $\mathbf{p}$.

Generalizing the treatment of rotating frames in GA1, the Lorentz rotation of a frame (79) can be related to the standard matrix form by writing

$$e_\mu = R\gamma_\mu\widetilde{R} = \alpha^\nu_\mu\gamma_\nu. \tag{89}$$

As in GA1, this can be solved for the matrix elements

$$\alpha^\nu_\mu = e_\mu\cdot\gamma^\nu = (\gamma^\nu R\gamma_\mu\widetilde{R})_{(0)}. \tag{90}$$

Or it can be solved for the rotor,[7] with the result

$$R = \pm(A\widetilde{A})^{-\frac{1}{2}}A, \tag{91}$$

where

$$A \equiv e_\mu\gamma^\mu = \alpha^\nu_\mu\gamma_\nu\gamma^\mu \tag{92}$$

Equation (89) can be used to describe a change of coordinate frames.

In the tensorial approach to Lorentz rotations, the coordinates $x^\mu = \gamma^\mu\cdot x$ of a point x transform according to

$$x^\mu \;\rightarrow\; x'^\mu = \alpha^\mu_\nu x^\nu, \quad \text{with} \quad \alpha^\mu_\nu\alpha^\nu_\lambda = \delta^\mu_\lambda \tag{93}$$

as the orthogonality condition on the transformation. This can be interpreted either as a *passive* or an *active* transformation. In the passive case, it is accompanied by a (usually implicit) transformation of coordinate frame:

$$\gamma_\mu \;\rightarrow\; \gamma'_\mu = \alpha_\mu^\lambda \gamma_\lambda, \tag{94}$$

so that each spacetime point $x = x^\mu \gamma_\mu = x'^\mu \gamma'_\mu$ is left unchanged.

In the active case, each spacetime point $x = x^\mu \gamma_\mu$ is mapped to a new spacetime point

$$x' = x'^\mu \gamma_\mu = x^\mu \gamma'_\mu = Rx\widetilde{R}, \tag{95}$$

where the last form was obtained by identifying γ'_μ with e_μ in (89). This shows that STA enables us to dispense with coordinates entirely in the treatment of Lorentz transformations. Consequently, we deal with active Lorentz transformations only in the coordinate-free form (64) or (61), and we dispense with passive transformations entirely.

If all this seems rather obvious, just turn to any textbook on relativistic quantum theory,[8] where the γ_μ are matrices and (89) is introduced as a change in matrix representation to prove relativistic invariance of the "Dirac operator" $\gamma^\mu \partial_\mu = \gamma'^\mu \partial'_\mu$. In STA this is recognized as a passive Lorentz transformation, so it is superfluous. Consequently, this aspect of Lorentz invariance need not be mentioned in our treatment of the Dirac equation in Section VII.

V. Spinor Particle Mechanics

Now we are prepared to exploit the unique advantages of STA with a spinor formulation of relativistic (or proper) mechanics. This approach has three major benefits. First, it articulates perfectly with the rotor formulation of nonrelativistic rigid body mechanics in GA1. Second, it articulates perfectly with Dirac's quantum theory of the electron, providing it with an informative and useful classical limit that includes a natural classical explanation for the gyromagnetic ratio $g = 2$. Indeed, the spinor used here for particle mechanics is an obvious special case of the real Dirac spinor introduced in Section VII. Finally, the spinor formulation simplifies the solution of problems in relativistic mechanics and automatically generalizes particle mechanics to include spin precession.

The rotor equation for a frame

$$e_\mu = R\gamma_\mu \widetilde{R} \tag{96}$$

can be used to describe the relativistic kinematics of a rigid body (with negligible dimensions) traversing a world line $x = x(\tau)$ with proper time τ, provided we identify e_0 with the proper velocity v of the body, so that

$$\frac{dx}{d\tau} = \dot{x} = v = e_0 = R\gamma_0 \widetilde{R}\,. \tag{97}$$

Then $\{e_\mu = e_\mu(\tau); \mu = 0, 1, 2, 3\}$ is a *comoving frame* traversing the world line along with the particle, and the rotor R must also be a function of proper time, so that, at each time τ, equation (96) describes a Lorentz rotation of some arbitrarily chosen fixed frame $\{\gamma_\mu\}$ into the comoving frame $\{e_\mu = e_\mu(\tau)\}$. Thus, we have a rotor-valued function of proper time $R = R(\tau)$ determining a 1-parameter family of Lorentz rotations $e_\mu(\tau) = R(\tau)\gamma_\mu\widetilde{R}(\tau)$. The rotor R is a *unimodular spinor*, as it satisfies the unimodular condition $R\widetilde{R} = 1$.

The spacelike vectors $e_k = R\gamma_k\widetilde{R}$ (for $k = 1, 2, 3$) can be identified with the principal axes of the body. But the same equations can be used for modeling a particle with an intrinsic angular momentum or *spin*, where e_3 is identified with the spin direction $\hat{s}$; so we write

$$\hat{s} = e_3 = R\gamma_3\widetilde{R}\,. \tag{98}$$

Later we see that this corresponds exactly to the spin vector in the Dirac theory where the magnitude of the spin has the constant value $|\,s\,| = \hbar/2$.

The *rotor equation of motion for* $R = R(\tau)$ has the form

$$\dot{R} = \tfrac{1}{2}\Omega R \tag{99}$$

where $\Omega = \Omega(\tau)$ is a bivector-valued function. The fact that $\Omega = 2\dot{R}\widetilde{R} = -\widetilde{\Omega}$ is necessarily a bivector is easily proved by differentiating $R\widetilde{R} = 1$. Differentiating (96) and using (99), we see that the equations of motion for the comoving frame have the form

$$\dot{e}_\mu = \Omega \cdot e_\mu\,. \tag{100}$$

Clearly Ω can be interpreted as a generalized *rotational velocity* of the comoving frame.

The dynamics of the rigid body, that is, the effect of external forces and torques on the body, is completely characterized by specifying Ω as a definite function of proper time. The single rotor equation (99) is equivalent to the set of four frame equations (100). Besides the theoretical advantage of being closely related to the Dirac equation, as we shall see, it has the practical advantage of being simpler and easier to solve than the set of frame equations (100). The corresponding nonrelativistic rotor equation for a spinning body was introduced in GA1. It should be noted that the nonrelativistic rotor equation describes only rotational motion, while its relativistic generalization (99) describes rotational and translational motion together.

For a classical particle with mass m and charge e in an electromagnetic field F, the dynamics is specified by

$$\Omega = \frac{e}{m}\,F\,. \tag{101}$$

So (100) gives the particle equation of motion

$$m\dot{v} = eF \cdot v \tag{102}$$

This may be recognized as the classical Lorentz force with tensor components $m\dot{v}^\mu = eF^{\mu\nu}v_\nu$, but note that tensor theory does not admit the more powerful rotor equation of motion (99).

As demonstrated in specific examples that follow, even if one is interested in the motion of a structureless point charge, the rotor equation (99) is easier to solve than the Lorentz force equation (102). However, if one wants to extend the model to an electron with spin, the same solution automatically describes the electron's spin precession. The result is physically meaningful too, for, as we see later, the classical model of an electron with proper rotational velocity (101) proportional to the field F gives the same gyromagnetic ratio as the Dirac equation. Indeed, it is a well-defined classical limit of the Dirac equation, though Planck's constant remains in the magnitude of the spin. This role of the electromagnetic field F as a rotational velocity is so simple and natural that it deserves a name. I propose to dub the relation (101) the *Lorentz Torque*, since it is a straightforward generalization of the Lorentz Force (102). It is noteworthy that this idea, which is so natural in STA, seems never to have occurred to physicists using tensor theory. This is one more example of the influence of mathematical language on physical theory.

A. Motion in constant electric and magnetic fields.

If F is a uniform field on spacetime, then $\dot{\Omega} = 0$ and (99) has the solution

$$R = e^{\frac{1}{2}\Omega\tau}R_0\,, \tag{103}$$

where $R_0 = R(0)$ specifies the initial conditions. When this is substituted into (103) we get the explicit τ dependence of the proper velocity v. The integration of (97) for the history $x(t)$ is most simply accomplished in the general case of arbitrary non-null F by exploiting the invariant decomposition $F = fe^{i\varphi}$ determined in (30). This separates Ω into mutually commuting parts $\Omega_1 = (e/m)f\cos\varphi$ and $\Omega_2 = (e/m)if\sin\varphi$, so

$$e^{\frac{1}{2}\Omega\tau} = e^{\frac{1}{2}(\Omega_1+\Omega_2)\tau} = e^{\frac{1}{2}\Omega_1\tau}e^{\frac{1}{2}\Omega_2\tau}\,. \tag{104}$$

It also determines an invariant decomposition of the initial velocity $v(0)$ into a component v_1 in the f-plane and a component v_2 orthogonal to the f-plane; thus,

$$v(0) = f^{-1}(f\cdot v(0)) + f^{-1}(f\wedge v(0)) = v_1 + v_2\,. \tag{105}$$

When this is substituted in (97) and (104) is used, we get

$$\frac{dx}{d\tau} = v = e^{\Omega_1\tau}v_1 + e^{\Omega_2\tau}v_2\,. \tag{106}$$

Note that this is an invariant decomposition of the motion into "electriclike" and "magneticlike" components. It integrates easily to give the history

$$x(\tau) - x(0) = 2(e^{\Omega_1\tau} - 1)\Omega_1^{-1}v_1 + 2(e^{\Omega_2\tau} - 1)\Omega_2^{-1}v_2\,. \tag{107}$$

This general result, which applies for arbitrary initial conditions and arbitrary uniform electric and magnetic fields, has such a simple form because it is expressed in terms of invariants. It looks far more complicated when subjected to a space-time split and expressed directly as a function of "laboratory fields" in an inertial system. Details are given in my mechanics book.[3]

B. Electron in the field of a plane wave.

As a second example with important applications, we integrate the rotor equation for a "classical test charge" in an electromagnetic plane wave.[9] This is useful for describing the interaction of electrons with lasers. As explained at the end of Section VI in GA1, any plane wave field $F = F(x)$ with proper propagation vector k can be written in the canonical form

$$F = fz\,, \tag{108}$$

where f is a constant null bivector ($f^2 = 0$), and the x-dependence of F is exhibited explicitly by

$$z(k\cdot x) = \alpha_+ e^{i(k\cdot x)} + \alpha_- e^{-i(k\cdot x)}\,, \tag{109}$$

with

$$\alpha_\pm = \rho_\pm e^{\pm i\delta_\pm}\,, \tag{110}$$

where $\delta_\pm$ and $\rho_\pm \geq 0$ are scalars. It is crucial to note that the "imaginary" i here is the unit pseudoscalar, because it endows these solutions with geometrical properties not possessed by conventional "complex solutions." Indeed, as noted in GA1, the pseudoscalar property of i implies that the two terms on the right side of (109) describe right and left circular polarizations. Thus, the orientation of i determines handedness of the solutions.

For the plane wave (108), Maxwell's equation reduces to the algebraic condition,

$$kf = 0\,. \tag{111}$$

This implies $k^2 = 0$ as well as $f^2 = 0$. To integrate the rotor equation of motion

$$\dot{R} = \frac{e}{2m}\,FR\,, \tag{112}$$

it is necessary to express F as a function of τ. This can be done by using special properties of F to find constants of motion. Multiplying (112) by k and using (111) we find immediately that kR is a constant of the motion. So, with the initial condition $R(0) = 1$, we obtain $k = kR = Rk = k\widetilde{R}$; whence

$$Rk\widetilde{R} = k\,. \tag{113}$$

Thus, the one parameter family of Lorentz rotations represented by $R = R(\tau)$ lies in the little group of the lightlike vector k. Multiplying (113) by (96), we find the constants of motion $k\cdot e_\mu = k\cdot\gamma_\mu$. This includes the constant

$$\omega = k\cdot v, \tag{114}$$

which can be interpreted as the frequency of the plane wave "seen by the particle." Since $v = dx/d\tau$, we can integrate (114) immediately to get

$$k \cdot (x(\tau) - x(0)) = \omega\tau \,. \tag{115}$$

Inserting this into (109) and absorbing $k \cdot x(0)$ in the phase factor, we get $z(k \cdot x) = z(\omega\tau)$, expressing the desired τ dependence of F. Equation (112) can now be integrated directly, with the result

$$R = \exp\left(ef z_1/2m\right) = 1 + \frac{e}{2m} f z_1 \,, \tag{116}$$

where

$$z_1 = \frac{2}{\omega} \sin\left(\omega\tau/2\right) \left[\alpha_+ e^{i\omega\tau/2} + \alpha_- e^{-i\omega\tau/2}\right] . \tag{117}$$

This gives the velocity v and, by integrating (97), the complete particle history. Details are given elsewhere.[9] It is of practical interest to know that this solution is equivalent to the "Volkov solution" of the Dirac equation for an electron in a plane wave field.[10] In this case, the quantum mechanical solution is equivalent to its classical limit. The solution has practical applications to the interaction of electrons with laser fields.[13]

The problem of motion in a Coulomb field has been solved by the same spinor method,[11] but no other exact solutions of the rotor equation (99) with Lorentz torque have been published.

C. Spin precession.

We have established that specification of kinematics by the rotor equation (99) and dynamics by $\Omega = (e/m)F$ is a geometrically perspicuous and analytically efficient means of characterizing the motion of a classical charged particle, and noted that it automatically provides us with a classical model of spin precession. Now let us take a more general approach to modeling and analyzing spin precession. Any dynamics of spin precession can be characterized by specifying a functional form for Ω. That includes gravitational precession[12] and electron spin precession in the Dirac theory. To facilitate the analysis for any given dynamical model, we first carry the analysis as far as possible for arbitrary Ω. Then we give a specific application to measurement of the g-factor for a Dirac particle.

The rotor equation of motion (99) determines both translational and rotational motions of the comoving frame (96), whatever the frame models physically. It is of interest to separate translational and rotational modes, though they are generally coupled. This can be done by a spacetime split by the particle velocity v or by the reference vector γ_0. We consider both ways and how they are related.

D. Larmor and Thomas precession.

To split the rotational velocity Ω by the velocity v, we write

$$\Omega = \Omega\, v^2 = (\Omega \cdot v)v + (\Omega \wedge v)v \,. \tag{118}$$

This produces the split

$$\Omega = \Omega_+ + \Omega_- \,, \tag{119}$$

where

$$\Omega_+ = \tfrac{1}{2}(\Omega + v\widetilde{\Omega}v) = (\Omega \cdot v)v = \dot{v}v \,, \tag{120}$$

and

$$\Omega_- = \tfrac{1}{2}(\Omega - v\widetilde{\Omega}v) = (\Omega \wedge v)v \,. \tag{121}$$

Note that $\Omega \cdot v = \dot{v}$ was used in (120) to express Ω_+ entirely in terms of the proper acceleration $\dot{v}$ and velocity v. This split has exactly the same form as the split (58) of the electromagnetic bivector into electric and magnetic parts corresponding here to Ω_+ and Ω_- respectively. However, it is a split with respect to the instantaneous "rest frame" of the particle rather than a fixed inertial frame. In the rest frame the relative velocity of the particle itself vanishes, of course, so the particle's acceleration is entirely determined by the "electriclike" part Ω_+, as (120) shows explicitly. The "magneticlike" part Ω_- is completely independent of the particle motion; it is the *Larmor precession* (frequency) of the spin for a particle with a magnetic moment, so let us refer to it as the Larmor precession in the general case.

Unfortunately, (119) does not completely decouple precession from translation because Ω_+ contributes to both. Also, we need a way to compare precessions at different points on the particle history. These difficulties can be resolved by adopting the γ_0-split

$$R = LU \,, \tag{122}$$

exactly as defined by (80) and subsequent equations. At every time τ, this split determines a "deboost" of relative vectors $e_k e_0 = R\gamma_k\gamma_0\widetilde{R} = R\boldsymbol{\sigma}_k\widetilde{R}$ $(k = 1, 2, 3)$ into relative vectors

$$\mathbf{e}_k = \widetilde{L}(e_k e_0)L = U\boldsymbol{\sigma}_k\widetilde{U} \tag{123}$$

in the fixed reference system of γ_0. The particle is brought to rest, so to speak, so we can watch it precess (or spin) in one place. The precession is described by an equation of the form

$$\frac{dU}{dt} = -\tfrac{1}{2}i\boldsymbol{\omega}U \,, \tag{124}$$

so, as already shown in GA1, differentiation of (123) yields the familiar equations for a rotating frame:

$$\frac{d\mathbf{e}_k}{dt} = \boldsymbol{\omega} \times \mathbf{e}_k \,. \tag{125}$$

The problem now is to express $\boldsymbol{\omega}$ in terms of the given Ω and determine the relative contributions of the parts Ω_+ and Ω_-. To do that, we use the time dilation factor $v_0 = v \cdot \gamma_0 = dt/d\tau$ to change the time variable in (124) and write

$$\omega = -i\boldsymbol{\omega} v_0 \tag{126}$$

so (124) becomes $\dot{U} = \frac{1}{2}\omega U$. Then differentiation of (122) and use of (99) gives

$$\Omega = 2\dot{R}\widetilde{R} = 2\dot{L}\widetilde{L} + L\omega\widetilde{L}\,. \tag{127}$$

Solving for ω and using the split (119), we get

$$\omega = \widetilde{L}\Omega_- L + \widetilde{L}\dot{v}vL - 2\widetilde{L}\dot{L}\,. \tag{128}$$

Differentiation of (87) leads to

$$\widetilde{L}(\dot{v}v)L = \widetilde{L}\dot{L} + \dot{L}\widetilde{L}\,, \tag{129}$$

while differentiation of (88) gives

$$2\dot{L}\widetilde{L} = \frac{\dot{v} \wedge (v + \gamma_0)}{1 + v \cdot \gamma_0}\,. \tag{130}$$

These terms combine to give the well-known *Thomas precession* frequency

$$\begin{aligned} \omega_T &= \big((2\dot{L}\widetilde{L}) \wedge \gamma_0\big)\gamma_0 = \dot{L}\widetilde{L} - \widetilde{L}\dot{L} \\ &= \frac{(\dot{v} \wedge v \wedge \gamma_0)\gamma_0}{1 + v \cdot \gamma_0} = i\left(\frac{v_0^2}{1 + v_0}\right)\mathbf{v} \times \dot{\mathbf{v}}\,. \end{aligned} \tag{131}$$

The last step here, expressing the proper vectors in terms of relative vectors, was derived from the split

$$\dot{v}v = \dot{v} \wedge v = v_0^2(\dot{\mathbf{v}} + i(\mathbf{v} \times \dot{\mathbf{v}}))\,. \tag{132}$$

Finally, writing

$$\omega_L = \widetilde{L}\,\Omega_- L \tag{133}$$

for the transformed Larmor precession, we have the desired result

$$\omega = \omega_T + \omega_L\,. \tag{134}$$

The Thomas term describes the effect of motion on the precession explicitly and completely.

E. The g-factor in spin precession.

Now let us apply the rotor approach to a practical problem of spin precession. In general, for a charged particle with an intrinsic magnetic moment in a *uniform* electromagnetic field $F = F_+ + F_-$,

$$\Omega = \frac{e}{mc}\big(F_+ + \frac{g}{2}F_-\big) = \frac{e}{mc}\left[\, F + \tfrac{1}{2}(g-2)F_- \,\right], \tag{135}$$

where as defined by (121), F_- is the magnetic field in the instantaneous rest frame of the particle, and g is the usual gyromagnetic ratio. This yields the classical equation of motion (102) for the velocity, but by (98) and (100) the equation of motion for the spin is

$$\dot{s} = \frac{e}{m}\left[\, F + \tfrac{1}{2}(g-2)F_- \,\right]\cdot s\,. \tag{136}$$

This is the well-known Bargmann-Michel-Telegdi (BMT) equation, which is used in high precision measurements of the g-factor for the electron and muon.

To apply the BMT equation, it must be solved for the rate of spin precession. The general solution for an arbitrary combination $F = \mathbf{E} + i\mathbf{B}$ of uniform electric and magnetic fields is most easily found by replacing the BMT equation by the rotor equation

$$\dot{R} = \frac{e}{2m}FR + R\tfrac{1}{2}(g-2)\left(\frac{e}{2m}\right)i\mathbf{B}_0\,, \tag{137}$$

where

$$i\mathbf{B}_0 = \widetilde{R}F_-R = \tfrac{1}{2}\big[\widetilde{R}FR - (\widetilde{R}FR)^\dagger\big]\,. \tag{138}$$

is an "effective magnetic field" in the "rest system" of the particle. With initial conditions $R(0) = L_0$, $U(0) = 1$, for a boost without spatial rotation, a solution of (137) is

$$R = \exp\left[\frac{e}{2m}\,F\tau\right]\; L_0\; \exp\left[\tfrac{1}{2}(g-2)\left(\frac{e}{2m}\right)i\mathbf{B}_0\tau\right]\,, \tag{139}$$

where $\mathbf{B}_0$ is defined by

$$\begin{aligned}\mathbf{B}_0 &= \frac{1}{2i}\big[\widetilde{L}_0FL_0 - (\widetilde{L}_0FL_0)^\dagger\big] \\ &= \mathbf{B} + \frac{v_{00}^2}{1+v_{00}}\mathbf{v}_0\times(\mathbf{B}\times\mathbf{v}_0) + v_{00}\mathbf{E}\times\mathbf{v}_0\,,\end{aligned} \tag{140}$$

where $v_{00} = v(0)\cdot\gamma_0 = (1-\mathbf{v}^2)^{-\frac{1}{2}}$. The first factor in (139) has the same effect on both the velocity v and the spin s, so the last factor gives directly the change in the relative directions of the relative velocity $\mathbf{v}$ and the spin $\mathbf{s}$. This can be measured experimentally.[3]

To conclude this section, some general remarks about the description of spin will be helpful in applications and in comparisons with more conventional approaches. We have represented the spin by the proper vector $s = |\,s\,|e_3$ defined

by (98) and alternatively by the relative vector $\boldsymbol{\sigma} = |\, s \,|\mathbf{e}_3$, where $\mathbf{e}_3$ is defined by (123). For a particle with proper velocity $v = L^2\gamma_0$, these two representations are related by

$$sv = L\boldsymbol{\sigma}\widetilde{L} \tag{141}$$

or, equivalently, by

$$\boldsymbol{\sigma} = \widetilde{L}(sv)L = \widetilde{L}\,sL\gamma_0\,. \tag{142}$$

A straightforward spacetime split of the proper spin vector s, like (48) for the velocity vector, gives

$$s\gamma_0 = s_0 + \mathbf{s}\,, \tag{143}$$

where

$$\mathbf{s} = s \wedge \gamma_0 \tag{144}$$

is the relative spin vector, and $s \cdot v = 0$ implies that

$$v_0 s_0 = \mathbf{v} \cdot \mathbf{s}\,. \tag{145}$$

From (141) and (143), the relation of $\mathbf{s}$ to $\boldsymbol{\sigma}$ is found to be

$$\mathbf{s} = \boldsymbol{\sigma} + (v_0 - 1)(\boldsymbol{\sigma} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\,, \tag{146}$$

where $v_0 = v \cdot \gamma_0$ and $\hat{\mathbf{v}} = \mathbf{v}/|\,\mathbf{v}\,|$. Both vectors $\mathbf{s}$ and $\boldsymbol{\sigma}$ are sometimes used in the literature, and some confusion results from a failure to recognize that they come from two different kinds of spacetime split. Of course either one can be used, since one determines the other, but $\boldsymbol{\sigma}$ is usually simpler because its magnitude is constant. Note from (146) that they are indistinguishable in the non-relativistic approximation.

VI. Electromagnetic Field Theory

In STA an electromagnetic field is represented by a bivector-valued function $F = F(x)$ on spacetime. The field produced by a source with proper current density $J = J(x)$ is determined by *Maxwell's Equation*

$$\nabla F = J\,. \tag{147}$$

As explained in Section II, the differential operator $\nabla = \partial_x$ in STA is regarded as the (vector) derivative with respect to a spacetime point x.

Since ∇ is a vector operator the expansion (10) applies, so we can write

$$\nabla F = \nabla \cdot F + \nabla \wedge F\,, \tag{148}$$

where $\nabla \cdot F$ is the *divergence* of F and $\nabla \wedge F$ is the *curl*. We can accordingly separate (147) into vector and trivector parts:

$$\nabla \cdot F = J\,, \tag{149}$$

$$\nabla \wedge F = 0. \tag{150}$$

This is the coordinate-free form for the two covariant tensor equations for the electromagnetic field in standard relativistic theory.

As a pedagogical point, it is worth noting that the decomposition (148) into divergence and curl is a straightforward generalization of the 3D vectorial decomposition introduced in GA1. Also note that, as standard SI units are not well suited for spacetime physics, we choose a system of units that minimizes the number of constants in basic equations. The reader can infer the choice from the spacetime split of Maxwell's equation given below.

The reduction of the two Maxwell equations (149) and (150) to the to a single "Maxwell's Equation" (147) brings many simplifications to electromagnetic theory. For example, the operator ∇ has an inverse so (147) can be solved for

$$F = \nabla^{-1} J\,, \tag{151}$$

Of course, ∇^{-1} is an integral operator that depends on boundary conditions on F for the region on which it is defined, so (151) is an integral form of Maxwell's equation. However, if the "current" $J = J(x)$ is the sole source of F, then (151) provides the unique solution to (147).

Next we survey other simplifications to the formulation and analysis of electromagnetic equations. Differentiating (147) we obtain

$$\nabla^2 F = \nabla J = \nabla \cdot J + \nabla \wedge J\,, \tag{152}$$

where ∇^2 is the d'Alembertian (35). Separately equating scalar and bivector parts of (152), we obtain the *charge conservation* law

$$\nabla \cdot J = 0 \tag{153}$$

and an alternative equation for the E-M field

$$\nabla^2 F = \nabla \wedge J\,. \tag{154}$$

A. Electromagnetic Potentials.

A different field equation is obtained by using the fact that, under general conditions, any continuous bivector field $F = F(x)$ can be expressed as a derivative with the specific form

$$F = \nabla(A + Bi)\,, \tag{155}$$

where $A = A(x)$ and $B = B(x)$ are vector fields, so F has a "vector potential" A and a "trivector potential" Bi. This is a generalization of the well-known

"Helmholtz theorem" in vector analysis.[4] Since $\bigtriangledown A = \bigtriangledown \cdot A + \bigtriangledown \wedge A$ with a similar equation for $\bigtriangledown B$, the bivector part of (155) can be written

$$F = \bigtriangledown \wedge A + (\bigtriangledown \wedge B)i\,, \tag{156}$$

while the scalar and pseudoscalar parts yield the so-called "Lorenz condition"

$$\bigtriangledown \cdot A = \bigtriangledown \cdot B = 0\,, \tag{157}$$

Inserting (155) into Maxwell's equation (147) and separating vector and trivector parts, we obtain the usual wave equation for the vector potential

$$\bigtriangledown^2 A = J\,, \tag{158}$$

as well as

$$\bigtriangledown^2 Bi = 0\,. \tag{159}$$

The last equation shows that B is independent of the source J, so it can be set to zero in (155). However, in a theory with magnetic charges, Maxwell's equation takes the form

$$\bigtriangledown F = J + iK, \tag{160}$$

where $K = K(x)$ is a vector field, the "magnetic current density." On substituting (155) into (160) we obtain in place of (159),

$$\bigtriangledown^2 Bi = iK\,. \tag{161}$$

The pseudoscalar i can be factored out to make (161) appear symmetrical with (157), but this symmetry between the roles of electric and magnetic currents is deceptive, because one is vectorial while the other is actually trivectorial.

The separation of the generalized Maxwell's equation (160) into parts with electric and magnetic sources can be achieved by again using (148) and again getting (149) for the vector part but getting

$$\bigtriangledown \wedge F = iK \tag{162}$$

for the trivector part. This equation can be made to look similar to (149) by duality to put it in the form

$$\bigtriangledown \cdot (Fi) = K\,. \tag{163}$$

Note that the dual Fi of the bivector F is also a bivector. Hereafter we restrict our attention to the "physical case" $K = 0$.

B. Maxwell's equation for material media.

Sometimes the source current J can be decomposed into a *conduction current* J^C and a magnetization current $\bigtriangledown \cdot M$, where the generalized *magnetization* $M = M(x)$ is a bivector field; thus

$$J = J^C + \bigtriangledown \cdot M\,. \tag{164}$$

The Gordon decomposition of the Dirac current is of this ilk. Because of the mathematical identity $\nabla \cdot (\nabla \cdot M) = (\nabla \wedge \nabla) \cdot M = 0$, the conservation law $\nabla \cdot J = 0$ implies also that $\nabla \cdot J^C = 0$. Using (164), equation (149) can be put in the form

$$\nabla \cdot G = J^C \tag{165}$$

where we have defined a new field

$$G = F - M\,. \tag{166}$$

A disadvantage of this approach is that it mixes up physically different kinds of entities, an E-M field F and a matter field M. However, in most materials M is a function of the field F, so when a "constitutive equation" $M = M(F)$ is known (165) becomes a well defined equation for F.

C. Energy-momentum tensor.

STA enables us to write the usual Maxwell energy-momentum tensor $T(n) = T(n(x), x)$ for the electromagnetic field in the compact form

$$T(n) = \tfrac{1}{2} F n \widetilde{F} = -\tfrac{1}{2} F n F\,. \tag{167}$$

Recall that the tensor field $T(n)$ is a vector-valued linear function on the tangent space at each spacetime point x describing the flow of energy-momentum through a surface with normal $n = n(x)$, By linearity $T(n) = n_\mu T^\mu$, where $n_\mu = n \cdot \gamma_\mu$ and

$$T^\mu \equiv T(\gamma^\mu) = \tfrac{1}{2} F \gamma^\mu \widetilde{F}\,. \tag{168}$$

The divergence of $T(n)$ can be evaluated by using Maxwell's equation (147), with the result

$$\partial_\mu T^\mu = T(\nabla) = J \cdot F\,. \tag{169}$$

Its value is the negative of the *Lorentz force* (density) $F \cdot J$, which is the rate of energy-momentum transfer from the source J to the field F.

D. Eigenvectors of the Maxwell Tensor.

The compact, invariant form (167) enables us to solve easily the eigenvector problem for the Maxwell energy-momentum tensor. If F is not a null field, it has the invariant decomposition $F = f e^{i\varphi}$ given by (30), which, when inserted in (167), gives

$$T(n) = -\tfrac{1}{2} f n f \tag{170}$$

This is simpler than (167) because f is simpler than F. Note also that it implies that all fields differing only by an arbitrary "duality factor" $e^{i\varphi}$ have the same energy-momentum tensor. The eigenvalues can be found from (170) by inspection. The bivector f determines a timelike plane. Any vector n in that plane satisfies $n \wedge f = 0$, or equivalently, $nf = -fn$. On the other hand, if n

is orthogonal to the plane, then $n \cdot f = 0$ and $nf = fn$. For these two cases, (170) gives us

$$T(n) = \pm\tfrac{1}{2} f^2 n \,. \tag{171}$$

Thus $T(n)$ has a pair of doubly degenerate eigenvalues $\pm\frac{1}{2}f^2$ corresponding to "eigenbivectors" f and if, all expressible in terms of F by inverting (30). This approach should be compared with conventional matrix methods to appreciate the simplifications achieved by STA.

E. Relation to tensor formulations.

The versatility of STA is also illustrated by the ease with which the above invariant formulation of "Maxwell theory" can be related to more conventional formulations. The tensor components $F^{\mu\nu}$ of the E-M field F are given by (17), whence, using (34), we find

$$\partial_\mu F^{\mu\nu} = J \cdot \gamma^\nu = J^\nu \tag{172}$$

for the tensor components of Maxwell's equation (149). Similarly, the tensor components of (163) are

$$\partial_{[\nu} F_{\alpha\beta]} = K^\mu \epsilon_{\mu\nu\alpha\beta} \,, \tag{173}$$

where the brackets indicate antisymmetrization and $\epsilon_{\mu\nu\alpha\beta} = i^{-1} \cdot (\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta)$. The tensor components of the energy-momentum tensor (168) are

$$\begin{aligned} T^{\mu\nu} = \gamma^\mu \cdot T^\nu &= -\tfrac{1}{2}(\gamma^\mu F \gamma^\nu F)_{(0)} \\ &= (\gamma^\mu \cdot F) \cdot (F \cdot \gamma^\nu) - \tfrac{1}{2}\gamma^\mu \cdot \gamma^\nu (F^2)_{(0)} \\ &= F^{\mu\alpha} F^\nu_\alpha - \tfrac{1}{2} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \end{aligned} \tag{174}$$

F. Spacetime splits in E-M theory.

To demonstrate how smoothly the proper formulation of E-M theory articulates with the relative formulation, we quickly survey several spacetime splits. A spacetime split of Maxwell's equation (147) puts it in the standard relative vector form for an inertial system. Thus, following the procedure in Section 4,

$$J\gamma_0 = J_0 + \mathbf{J} \tag{175}$$

splits the current J into a charge density $J_0 = J \cdot \gamma_0$ and a relative current $\mathbf{J} = J \wedge \gamma_0$ in the γ_0-system. Similarly,

$$\gamma_0 \nabla = \partial_t + \boldsymbol{\nabla} \tag{176}$$

splits $\nabla = \partial_x$ into a time derivative $\partial_t = \gamma_0 \cdot \nabla$ and spatial derivative $\boldsymbol{\nabla} = \gamma_0 \wedge \nabla = \partial_{\mathbf{x}}$ with respect to the relative position vector $\mathbf{x} = x \wedge \gamma_0$. Combining this with the split of F into electric and magnetic parts, we get Maxwell's equation (147) in the split form

$$(\partial_t + \boldsymbol{\nabla})(\mathbf{E} + i\mathbf{B}) = J_0 - \mathbf{J} \,, \tag{177}$$

in agreement with the formulation in GA1.

Note that (176) splits the D'Alembertian into

$$\nabla^2 = (\nabla\gamma_0)(\gamma_0\nabla) = (\partial_t - \boldsymbol{\nabla})(\partial_t + \boldsymbol{\nabla}) = \partial_t^2 - \boldsymbol{\nabla}^2. \tag{178}$$

The vector field $T^0 = T(\gamma^0) = T(\gamma_0)$ is the *energy-momentum density* in the γ_0-system. The split

$$T^0\gamma^0 = T^0\gamma_0 = T^{00} + \mathbf{T}^0 \tag{179}$$

separates it into an energy density $T^{00} = T^0 \cdot \gamma^0$ and a momentum density $\mathbf{T}^0 = T^0 \wedge \gamma^0$. Using the fact that γ_0 anticommutes with relative vectors, from (168) we obtain

$$T^0\gamma^0 = \tfrac{1}{2}FF^\dagger = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \mathbf{E}\times\mathbf{B}\,, \tag{180}$$

in agreement with GA1.

The spacetime split helps us with physical interpretation. Corresponding to the split $F = \mathbf{E} + i\mathbf{B}$, the magnetization field M splits into

$$M = -\mathbf{P} + i\mathbf{M}\,, \tag{181}$$

where $\mathbf{P}$ is the electric *polarization density* and $\mathbf{M}$ is the *magnetic moment density*. Writing

$$G = \mathbf{D} + i\mathbf{H}\,, \tag{182}$$

we see that (166) gives us the familiar relations

$$\mathbf{D} = \mathbf{E} + \mathbf{P}\,, \tag{183}$$

$$\mathbf{H} = \mathbf{B} - \mathbf{M}\,. \tag{184}$$

Insertion of (182) into (165) with a spacetime split yields the usual set of Maxwell's equations for a material medium.

VII. Real Relativistic Quantum Theory

The Dirac equation is the cornerstone of relativistic quantum theory, if not the single most important equation in all of quantum physics. This Section shows how STA simplifies the entire Dirac theory, reveals hidden geometric structure with implications for physical interpretation, and provides a common spinor method for classical and quantum physics with a more direct and transparent classical limit of the Dirac equation.

First, we show how to reformulate the standard matrix version of Dirac theory in terms of the real STA. As this reformulation eliminates superfluous complex numbers and matrices from the standard version, I call it the *real Dirac theory*.

Next we provide the real Dirac wave function with a geometric interpretation by relating it to *local observables*. The term "local observable" is non-standard but the concept is not unprecedented. It refers to assignment of physical interpretation to some local quantity such as energy or charge density rather than to global quantities such as expectation values. It serves as a device for describing local geometric structure of the theory quite apart from claims of objective reality. Its bearing on the interpretation of quantum mechanics is discussed in the next Section.

For reference purposes, I provide a complete catalog of relations between local observables in the real theory and the socalled "bilinear covariants" in the matrix theory. This facilitates translation between the two formulations. It will be noted that the real version is substantially simpler, and the complexities of translation can be avoided by sticking to the real theory alone.

Finally, I provide a thorough analysis of local conservation laws in the real Dirac theory to ascertain further what STA can tell us about geometric structure and physical interpretation. The analysis is much more complete than any treatment in textbooks that I know.

This account is limited to the single particle Dirac theory. The tendency in textbooks is to forego a thorough study of single particle theory and leap at once to the second quantized many particle theory. I leave it to the reader to decide what might be lost by that practice.

Space does not permit an adequate account of "real solutions" of the Dirac equation in this article. Partial treatments are given elsewhere,[15,16] but it is worth mentioning here that in some respects the real Dirac equation is easier to solve and analyze than the Schroedinger equation.

A. Derivation of the real Dirac theory.

Derivation of the real STA version of the Dirac theory from the standard matrix version is essentially the same as for the Pauli theory, but the differences are sufficient to justify a quick review. To find a representation of the Dirac theory in terms of STA, we begin with a Dirac spinor Ψ, a column matrix of 4 complex numbers. Let u be a fixed spinor with the properties

$$u^\dagger u = 1\,, \tag{185}$$

$$\gamma_0 u = u\,, \tag{186}$$

$$\gamma_2\gamma_1 u = i'u\,. \tag{187}$$

In writing this we regard the γ_μ, for the moment, as 4×4 Dirac matrices, and i' as the unit imaginary in the complex number field of the Dirac algebra. Now, we can write any Dirac spinor

$$\Psi = \psi u\,, \tag{188}$$

where ψ is a matrix that can be expressed as a polynomial in the γ_μ. The coefficients in this polynomial can be taken as real, for if there is a term with an imaginary coefficient, then (187) enables us to make it real without altering

(188) by replacing i' in the term by $\gamma_2\gamma_1$ on the right of the term. Furthermore, the polynomial can be taken to be an even multivector, for if any term is odd, then (186) allows us to make it even by multiplying on the right by γ_0. Thus, in (188) we may assume that ψ *is a real even multivector*, so we can reinterpret the γ_μ in ψ as vectors in STA instead of matrices. Thus, we have established a correspondence between Dirac spinors and even multivectors in STA. The correspondence must be one-to-one, because the space of even multivectors (like the space of Dirac spinors) is exactly 8-dimensional, with 1 scalar, 1 pseudoscalar and 6 bivector dimensions.

Finally, it should be noted that by eliminating the ungeometrical imaginary i' from the base field we reduce the degrees of freedom in the Dirac theory by half, with consequent simplification of the theory that shows up in the real version. The *Dirac algebra* is generated by the Dirac matrices over the base field of complex numbers, so it has $2^4 \times 2 = 32$ degrees of freedom and can be identified with the algebra of 4×4 complex matrices. From (14) we see that STA has $2^4 = 16$ degrees of freedom.

One immediate simplification brought by STA appears in the spacetime split. To write his equation in hamiltonian form, Dirac defined 4×4 matrices

$$\alpha_k = \gamma_k\gamma_0 \tag{189}$$

for $k = 1, 2, 3$. This is, in fact, a representation of the 2×2 Pauli matrices by 4×4 matrices. STA eliminates this awkward and irrelevant distinction between matrix representations of different dimension, so the α_k can be identified with the $\boldsymbol{\sigma}_k$, as we have already done in the spacetime split (43).

There are several ways to represent a Dirac spinor in STA,[18] but all representations are, of course, mathematically equivalent. The representation chosen here has the advantages of simplicity and, as we shall see, ease of interpretation.

To distinguish a spinor ψ in STA from its matrix representation Ψ in the Dirac algebra, let us call it a *real spinor* to emphasize the elimination of the ungeometrical imaginary i'. Alternatively, we might refer to ψ as the *operator representation* of a Dirac spinor, because, as shown below, it plays the role of an operator generating observables in the theory.

In terms of the real wave function ψ, the Dirac equation for an electron can be written in the form

$$\gamma^\mu(\partial_\mu\psi\gamma_2\gamma_1\hbar - eA_\mu\psi) = m\psi\gamma_0\,, \tag{190}$$

where m is the mass and $e = -|\,e\,|$ is the charge of the electron, while the $A_\mu = A \cdot \gamma_\mu$ are components of the electromagnetic vector potential. To prove that this is equivalent to the standard matrix form of the Dirac equation,[8] we simply interpret the γ_μ as matrices, multiply by u on the right and use (186) and (188) to get the standard form

$$\gamma^\mu(i'\hbar\partial_\mu - eA_\mu)\Psi = m\Psi\,. \tag{191}$$

This completes the proof. Alternative proofs are given elsewhere.[17–19] The original converse derivation of (190) from (191) was much more indirect.[14]

Henceforth, we can work with the *real Dirac equation* (190) without reference to its matrix representation (191). We know from previous Sections that computations in STA can be carried out without introducing a basis, and we recognize the so-called "Dirac operator" $\nabla = \gamma^\mu \partial_\mu$ as the vector derivative with respect to a spacetime point, so let us write the real Dirac equation in the coordinate-free form

$$\nabla\psi\mathbf{i}\hbar - eA\psi = m\psi\gamma_0\,, \tag{192}$$

where $A = A_\mu\gamma^\mu$ is the electromagnetic vector potential, and the notation

$$\mathbf{i} \equiv \gamma_2\gamma_1 = i\gamma_3\gamma_0 = i\boldsymbol{\sigma}_3 \tag{193}$$

emphasizes that this bivector plays the role of the imaginary i' that appears explicitly in the matrix form (191) of the Dirac equation. To interpret the theory, it is crucial to note that the bivector $\mathbf{i}$ has a definite geometrical interpretation while i' does not.

B. Lorentz invariance.

Equation (192) is Lorentz invariant, despite the explicit appearance of the constants γ_0 and $\mathbf{i} = \gamma_2\gamma_1$ in it. These constants need not be associated with vectors in a particular reference frame, though it is often convenient to do so. It is only required that γ_0 be a fixed, future-pointing, timelike unit vector while $\mathbf{i}$ is a spacelike unit bivector that commutes with γ_0. The constants can be changed by a Lorentz rotation

$$\gamma_\mu \to \gamma'_\mu = C\gamma_\mu\widetilde{C}\,, \tag{194}$$

where C is a constant rotor, so $C\widetilde{C} = 1$,

$$\gamma'_0 = C\gamma_0\widetilde{C} \qquad \text{and} \qquad \mathbf{i}' = C\mathbf{i}\widetilde{C}\,. \tag{195}$$

A corresponding change in the wave function,

$$\psi \to \psi' = \psi\widetilde{C}\,, \tag{196}$$

induces a mapping of the Dirac equation (192) into an equation of the same form:

$$\nabla\psi\mathbf{i}'\hbar - eA\psi' = m\psi'\gamma'_0\,. \tag{197}$$

This transformation is no more than a change of constants in the Dirac equation. It need not be coupled to a change in reference frame. Indeed, in the matrix formulation it can be interpreted as a mere change in matrix representation, that is, a change in the particular matrices selected to be associated with the vectors γ_μ, for (188) gives

$$\Psi = \psi u = \psi' u'\,, \tag{198}$$

where $u' = Cu$.

For the special case

$$C = e^{\mathbf{i}\varphi_0}\,, \tag{199}$$

where φ_0 is a scalar constant, (195) gives $\gamma_0' = \gamma_0$ and $\mathbf{i}' = \mathbf{i}$, so ψ and

$$\psi' = \psi e^{\mathbf{i}\varphi_0} \tag{200}$$

are solutions of the same equation. In other words, the Dirac equation does not distinguish solutions differing by a constant phase factor.

C. Charge conjugation.

Note that $\boldsymbol{\sigma}_2 = \gamma_2\gamma_0$ anticommutes with both γ_0 and $\mathbf{i} = i\boldsymbol{\sigma}_3$, so multiplication of the Dirac equation (192) on the right by $\boldsymbol{\sigma}_2$ yields

$$\nabla\psi^C\mathbf{i}\hbar + eA\psi^C = m\psi^C\gamma_0\,, \tag{201}$$

where

$$\psi^C = \psi\boldsymbol{\sigma}_2\,. \tag{202}$$

The net effect is to change the sign of the charge in the Dirac equation, therefore, the transformation $\psi \to \psi^C$ can be interpreted as *charge conjugation*. Of course, the definition of charge conjugate is arbitrary up to a constant phase factor such as in (200). The main thing to notice here is that in (202) charge conjugation, like parity conjugation, is formulated as a completely geometrical transformation, without any reference to a complex conjugation operation of obscure physical meaning. Its geometric meaning is determined by what it does to the "frame of observables" identified below.

D. Interpretation of the Dirac wave function.

As explained in Section II, since the real Dirac wave function $\psi = \psi(x)$ is an even multivector, we can write

$$\psi\widetilde{\psi} = \rho e^{i\beta}\,, \tag{203}$$

where ρ and β are scalars. Hence ψ has the *Lorentz invariant decomposition*

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}}R, \quad \text{where} \quad R\widetilde{R} = \widetilde{R}R = 1\,. \tag{204}$$

At each spacetime point x, the rotor $R = R(x)$ determines a Lorentz rotation of a given fixed frame of vectors $\{\gamma_\mu\}$ into a frame $\{e_\mu = e_\mu(x)\}$ given by

$$e_\mu = R\gamma_\mu\widetilde{R}\,. \tag{205}$$

In other words, R determines a unique frame field on spacetime.

The physical interpretation given to the frame field $\{e_\mu\}$ is a key to the interpretation of the entire Dirac theory. Specifically, the e_μ can be interpreted directly as descriptors of the kinematics of electron motion. It follows from (205), therefore, that the rotor field $R = R(x)$ is a descriptor of electron kinematics.

It should be noted that (205) has the same algebraic form as the *comoving frame* (96) defined on classical partical histories. Thus, (205) is a direct generalization of (96) from frames on curves to frame fields on spacetime. Conversely, as we shall see, probability conservation in the Dirac theory permits a decomposition of the frame field into bundles of comoving frames on Dirac "streamlines." This provides a direct connection to the classical spinor particle mechanics in Section V and thereby a natural approach to the classical limit of the Dirac equation, as discussed in the next Section.

Anticipating that the factor $(\rho e^{i\beta})^{\frac{1}{2}}$ can be given a statistical interpretation, the canonical form (204) can be regarded as an invariant decomposition of the Dirac wave function into a 2-parameter *statistical factor* $(\rho e^{i\beta})^{\frac{1}{2}}$ and a 6-parameter *kinematical factor* R.

From (204), (205) and (196) we find that

$$\psi\gamma_\mu\widetilde{\psi} = \psi'\gamma'_\mu\widetilde{\psi}' = \rho e_\mu \,. \tag{206}$$

Note that we have here a set of four linearly independent vector fields which are invariant under the transformation specified by (194) and (195). Thus these fields do not depend on any coordinate system, despite the appearance of γ_μ on the left side of (206). Note also that the factor $e^{i\beta/2}$ in (204) does not contribute to (206), because the pseudoscalar i anticommutes with the γ_μ.

Two of the vector fields in (206) are given physical interpretations in the standard Dirac theory. First, the vector field

$$\psi\gamma_0\widetilde{\psi} = \rho e_0 = \rho v \tag{207}$$

is the *Dirac current*, which, in accord with the standard *Born interpretation*, we interpret as a *probability current*. Thus, at each spacetime point x the timelike vector $v = v(x) = e_0(x)$ is interpreted as the *probable* (proper) *velocity* of the electron, and $\rho = \rho(x)$ is the relative probability (i.e. *proper probability density*) that the electron actually is at x. The correspondence of (207) to the conventional definition of the Dirac current is displayed in Table I.

The probability conservation law

$$\nabla \cdot (\psi\gamma_0\widetilde{\psi}) = \nabla \cdot (\rho v) = 0 \tag{208}$$

follows directly from the Dirac equation. To prove that we can use (204) and (205) to put the Dirac equation (192) into the form

$$\hbar(\nabla\psi)\gamma_0\widetilde{\psi} = m\rho e^{i\beta}e_1e_2 + \rho A e_1e_2e_0, \tag{209}$$

from which it follows that

$$\begin{aligned}[(\nabla\psi)\gamma_0\widetilde{\psi}]_{(0)} &= \tfrac{1}{2}[(\nabla\psi)\gamma_0\widetilde{\psi} + \psi\gamma_0(\nabla\psi)\tilde{}\,]_{(0)} \\ &= \tfrac{1}{2}\gamma^\mu \cdot (\partial_\mu\psi\gamma_0\widetilde{\psi} + \psi\gamma_0\partial_\mu\widetilde{\psi}) = 0.\end{aligned} \tag{210}$$

The vector field

$$\tfrac{1}{2}\hbar\psi\gamma_3\widetilde{\psi} = \rho\tfrac{1}{2}\hbar e_3 = \rho s \tag{211}$$

TABLE I: BILINEAR COVARIANTS

Scalar	$\widetilde{\Psi}\Psi = \Psi^\dagger\gamma_0\Psi = (\psi\widetilde{\psi})_{(0)} = \rho\cos\beta$
Vector	$\widetilde{\Psi}\gamma_\mu\Psi = \Psi^\dagger\gamma_0\gamma_\mu\Psi = (\psi\gamma_0\widetilde{\psi}\gamma_\mu)_{(0)} = (\psi^\dagger\gamma_0\gamma_\mu\psi)_{(0)}$ $= (\psi\gamma_0\widetilde{\psi})\cdot\gamma_\mu = (\rho v)\cdot\gamma_\mu = \rho v_\mu$
Bivector	$\frac{e}{m}\,\frac{i'\hbar}{2}\,\widetilde{\Psi}\frac{1}{2}\big(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\big)\Psi = \frac{e\hbar}{2m}\,\big(\gamma_\mu\gamma_\nu\psi\gamma_2\gamma_1\widetilde{\psi}\big)_{(0)}$ $= (\gamma_\mu\wedge\gamma_\nu)\cdot(M) = M_{\nu\mu} = \frac{e}{m}\,\rho(ie^{i\beta}sv)\cdot(\gamma_\mu\wedge\gamma_\nu)$
Pseudovector*	$\frac{1}{2}i'\hbar\widetilde{\Psi}\gamma_\mu\gamma_5\Psi = \frac{1}{2}\hbar(\gamma_\mu\psi\gamma_3\widetilde{\psi})_{(0)} = \gamma_\mu\cdot(\rho s) = \rho s_\mu$
Pseudoscalar*	$\widetilde{\Psi}\gamma_5\Psi = (i\psi\widetilde{\psi})_{(0)} = -\rho\sin\beta$

*Here we use the more conventional symbol $\gamma_5{=}\gamma_0\gamma_1\gamma_2\gamma_3$ for the matrix representation of the unit pseudoscalar i.

will be interpreted as the *spin vector* density, in exact correspondence with the real PS theory. Justification for this interpretation comes from angular momentum conservation treated below. Note in Table I that this vector quantity is represented as a pseudovector (or axial vector) quantity in the conventional matrix formulation. The spin pseudovector is correctly identified as is, as shown below.

As we have noted before, angular momentum is actually a bivector quantity. The *spin angular momentum* $S = S(x)$ is a bivector field related to the spin vector field $s = s(x)$ by

$$S = isv = \tfrac{1}{2}\hbar ie_3e_0 = \tfrac{1}{2}\hbar R\gamma_2\gamma_1\widetilde{R} = \tfrac{1}{2}R(\mathbf{i}\hbar)\widetilde{R}. \tag{212}$$

The right side of this chain of equivalent representations shows the relation of the spin to the unit imaginary $\mathbf{i}$ appearing in the Dirac equation (192). Indeed, it shows that *the bivector* $\frac{1}{2}\mathbf{i}\hbar$ is a *reference representation* of the spin that is rotated by the kinematical factor R into the local spin direction at each space-time point. This establishes an explicit connection between spin and imaginary numbers that is inherent in the Dirac theory but hidden in the conventional formulation, and, as we have already seen, remains even in the Schroedinger approximation.

Explicit equations relating spin to the unit imaginary i' in the PS theory are given in GA1. They apply without change in the Dirac theory, so the argument

need not be repeated here. The important fact is that for every solution of the Dirac equation, at each spacetime point x the bivector $S = S(x)$ specifies a definite spacelike tangent plane, a *spin plane*, if you will.

Explicit identification of S with spin is not made in standard accounts of the Dirac theory.[8] Typically, they introduce the spin (density) tensor

$$\rho S^{\nu\alpha\beta} = \frac{i'\hbar}{2}\,\widetilde{\Psi}\gamma^\nu \wedge \gamma^\alpha \wedge \gamma^\beta \Psi = \frac{i'\hbar}{2}\,\widetilde{\Psi}\gamma_\mu\gamma_5\Psi\epsilon^{\mu\nu\alpha\beta} = \rho s_\mu \epsilon^{\mu\nu\alpha\beta}\,, \tag{213}$$

where use has been made of the identity

$$\gamma^\nu \wedge \gamma^\alpha \wedge \gamma^\beta = \gamma_\mu\gamma_5\epsilon^{\mu\nu\alpha\beta} \tag{214}$$

and the expression for s_μ in Table I. Note that the "alternating tensor" $\epsilon^{\mu\nu\alpha\beta}$ can be *defined* simply as the product of two pseudoscalars, thus

$$\begin{aligned}\epsilon^{\mu\nu\alpha\beta} &= -i(\gamma^\mu \wedge \gamma^\nu \wedge \gamma^\alpha \wedge \gamma^\beta) = -(i\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta)_{(0)} \\ &= -(\gamma_3 \wedge \gamma_2 \wedge \gamma_1 \wedge \gamma_0)\cdot(\gamma^\mu \wedge \gamma^\nu \wedge \gamma^\alpha \wedge \gamma^\beta)\,.\end{aligned} \tag{215}$$

$$\gamma^\mu \wedge \gamma^\nu \wedge \gamma^\alpha \wedge \gamma^\beta = i\epsilon^{\mu\nu\alpha\beta}\,. \tag{216}$$

From (213) and (215) we find

$$S^{\nu\alpha\beta} = s_\mu\epsilon^{\mu\nu\alpha\beta} = -i(s \wedge \gamma^\nu \wedge \gamma^\alpha \wedge \gamma^\beta) = -(is)\cdot(\gamma^\nu \wedge \gamma^\alpha \wedge \gamma^\beta)\,. \tag{217}$$

The last expression shows that the $S^{\nu\alpha\beta}$ are simply tensor components of the pseudovector is. Contraction of (217) with $v_\nu = v\cdot\gamma_\nu$ and use of duality gives the desired relation between $S^{\nu\alpha\beta}$ and S:

$$\begin{aligned}v_\nu S^{\nu\alpha\beta} &= -i(s \wedge v \wedge \gamma^\alpha \wedge \gamma^\beta) = -[\,i(s\wedge v)\,]\cdot(\gamma^\alpha \wedge \gamma^\beta) \\ &= S\cdot(\gamma^\beta \wedge \gamma^\alpha) = S^{\alpha\beta}\,.\end{aligned} \tag{218}$$

Its significance will be made clear in the discussion of angular momentum conservation.

Note that the spin bivector and its relation to the unit imaginary is invisible in the standard version of the bilinear covariants in Table I. The spin S is buried there in the magnetization (tensor or bivector). The magnetization M can be defined and related to the spin by

$$M = \frac{e\hbar}{2m}\,\psi\gamma_2\gamma_1\widetilde{\psi} = \frac{e\hbar}{2m}\,\rho e^{i\beta}e_2e_1 = \frac{e}{m}\,\rho S e^{i\beta}\,. \tag{219}$$

One source for the interpretation of M as magnetization is the Gordon decomposition of the Dirac current given below. Equation (219) reveals that in the Dirac theory the magnetic moment is not simply proportional to the spin as often asserted; the two are related by a *duality rotation* produced by the factor $e^{i\beta}$. It may be appreciated that this relation of M to S is much simpler than any relation of $M^{\alpha\beta}$ to $S^{\nu\alpha\beta}$ in the literature, another indication that S is the most appropriate representation for spin. By the way, note that (219) provides some

justification for referring to β henceforth as the *duality parameter*. The name is noncommittal to the physical interpretation of β, a debatable issue discussed later.

We are now better able to assess the content of Table I. There are $1+4+6+4+1 = 16$ distinct bilinear covariants but only 8 parameters in the wave function, so the various covariants are not mutually independent. Their interdependence has been expressed in the literature by a system of algebraic relations known as "Fierz Identities."[20] However, the invariant decomposition of the wave function (204) reduces the relations to their simplest common terms. Table I shows exactly how the covariants are related by expressing them in terms of ρ, β, v_μ, s_μ, which constitutes a set of 7 independent parameters, since the velocity and spin vectors are constrained by the three conditions that they are orthogonal and have constant magnitudes. This parametrization reduces the derivation of any Fierz identity practically to inspection. Note, for example, that

$$\rho^2 = (\widetilde{\Psi}\Psi)^2 + (\widetilde{\Psi}\gamma_5\Psi)^2 = (\widetilde{\Psi}\gamma_\mu\Psi)(\widetilde{\Psi}\gamma^\mu\Psi) = -(\widetilde{\Psi}\gamma_\mu\gamma_5\Psi)(\widetilde{\Psi}\gamma^\mu\gamma_5\Psi)\,. \quad (220)$$

Evidently Table I tells us all we need to know about the bilinear covariants and makes further reference to Fierz identities superfluous.

Note that the factor $i'\hbar$ occurs explicitly in Table I only in those expressions involving electron spin. The conventional justification for including the i' is to make antihermitian operators hermitian so the bilinear covariants are real. We have seen however that this smuggles spin into the expressions. That can be made explicit by using (212) to derive the general identity

$$i'\hbar\widetilde{\Psi}\Gamma\Psi = \widetilde{\Psi}\Gamma\gamma_\alpha\gamma_\beta\Psi S^{\alpha\beta}\,, \quad (221)$$

where Γ is any matrix operator.

Perhaps the most significant thing to note about Table I is that only 7 of the 8 parameters in the wave function are involved. The missing parameter is the *phase* of the wave function. To understand the significance of this, note also that, in contrast to the vectors e_0 and e_3 representing velocity and spin directions, the vectors e_1 and e_2 do not appear in Table I except indirectly in the product e_2e_1. The missing parameter is one of the six parameters implicit in the rotor R determining the Lorentz rotation (205). We have already noted that 5 of these parameters are needed to determine the velocity and spin directions e_0 and e_3. By duality, these vectors also determine the direction $e_2e_1 = ie_3e_0$ of the "spin plane" containing e_1 and e_2. The remaining parameter therefore determines the directions of e_1 and e_2 in this plane. It is literally an angle of rotation in this plane and the spin bivector $\hat{S} = e_2e_1 = R\mathbf{i}\widetilde{R}$ is the generator of the rotation. Thus, in full accord with PS theory we arrive at a *geometrical interpretation of the phase of the wave function* that is *inherent in the Dirac theory*. But all of this is invisible in the conventional matrix formulation.

The purpose of Table I is to explicate the correspondence of the matrix formulation to the real (STA) formulation of the Dirac theory. Once it is understood that the two formulations are completely isomorphic, the matrix formulation

can be dispensed with and Table I becomes superfluous. By revealing the geometrical meaning of the unit imaginary and the wave function phase along with this connection to spin, STA challenges us to ascertain the physical significance of these geometrical facts.

E. Conservation laws.

One of the miracles of the Dirac theory was the spontaneous emergence of spin in the theory when nothing about spin seemed to be included in the assumptions. This miracle has been attributed to Dirac's derivation of his linearized relativistic wave equation, so spin has been said to be "a relativistic phenomenon." However, we have seen that the "Dirac operator" $\nabla = \gamma^\mu \partial_\mu$ is a generic spacetime derivative equally suited to the formulation of Maxwell's equation, and we have concluded that the Dirac algebra arises from spacetime geometry rather than anything special about quantum theory. The origin of spin must be elsewhere.

Our ultimate objective is to ascertain precisely what features of the Dirac theory are responsible for its extraordinary empirical success and to establish a coherent physical interpretation that accounts for all its salient aspects. The geometric insights of STA provide us with a perspective from which to criticize some conventional beliefs about quantum mechanics and so leads us to some unconventional conclusions. However, our purpose here is merely to raise significant issues by introducing suggestive interpretations. Much more will be required to claim definitive conclusions.

The physical interpretation of standard quantum mechanics is centered on meaning ascribed to the *kinetic energy-momentum operators* $\underline{p}_\mu$ defined in the conventional matrix theory by

$$\underline{p}_\mu = i'\hbar\partial_\mu - e\,A_\mu\,. \tag{222}$$

In the STA formulation they are defined by

$$\underline{p}_\mu = \underline{i}\hbar\partial_\mu - e\,A_\mu\,, \tag{223}$$

where the underbar signifies a "linear operator" and the operator $\underline{i}$ signifies right multiplication by the bivector $\mathbf{i} = \gamma_2\gamma_1$, as defined by

$$\underline{i}\psi = \psi\mathbf{i}\,. \tag{224}$$

The importance of (223) can hardly be overemphasized. Above all, it embodies the fruitful "minimal coupling" rule, a fundamental principle of gauge theory that fixes the form of electromagnetic interactions. In this capacity it plays a crucial heuristic role in the original formulation of the Dirac equation, as is clear when the equation is written in the form

$$\gamma^\mu \underline{p}_\mu\psi = \psi\gamma_0 m\,. \tag{225}$$

However, the STA formulation tells us even more. It reveals geometrical properties of the $\underline{p}_\mu$ that provide clues to a deeper physical meaning. We have already noted a connection of the factor $\mathbf{i}\hbar$ with spin. We establish below that this connection is a consequence of the form and interpretation of the $\underline{p}_\mu$. Thus,

TABLE II: Observables of the energy-momentum operator, relating real and matrix versions.

Energy-momentum tensor	$T^{\mu\nu} = T^\mu \cdot \gamma^\nu = (\gamma_0 \widetilde{\psi} \gamma^\mu \underline{p}^\nu \psi)_{(0)}$ $= \widetilde{\Psi} \gamma^\mu \underline{p}^\nu \Psi$
Kinetic energy density	$T^{00} = (\psi^\dagger \underline{p}^0 \psi)_{(0)} = \Psi^\dagger \underline{p}^0 \Psi$
Kinetic momentum density	$T^{0k} = (\psi^\dagger \underline{p}^k \psi)_{(0)} = \Psi^\dagger \underline{p}^k \Psi$
Angular Momentum tensor	$J^{\nu\alpha\beta} = \left[T^\nu \wedge x + i\rho(s \wedge \gamma^\nu) \right] \cdot (\gamma^\beta \wedge \gamma^\alpha)$ $= T^{\nu\alpha} x^\beta - T^{\nu\beta} x^\alpha - \frac{i'\hbar}{2} \widetilde{\Psi} \gamma_5 \gamma_\mu \Psi \epsilon^{\mu\nu\alpha\beta}$
Gordon current	$K_\mu = \frac{e}{m} (\widetilde{\psi} \underline{p}_\mu \psi)_{(0)} = \frac{e}{m} \widetilde{\Psi} \underline{p}_\mu \Psi$

spin was inadvertently smuggled into the Dirac theory by the $\underline{p}_\mu$, hidden in the innocent looking factor $i'\hbar$. Its sudden appearance was only incidentally related to relativity. History has shown that it is impossible to recognize this fact in the conventional formulation of the Dirac theory. The connection of $i'\hbar$ with spin is not inherent in the $\underline{p}_\mu$ alone. It appears only when the $\underline{p}_\mu$ operate on the wave function, as is evident from (212). This leads to the conclusion that the significance of the $\underline{p}_\mu$ lies in what they imply about the physical meaning of the wave function. Indeed, the STA formulation reveals that the $\underline{p}_\mu$ have something important to tell us about the kinematics of electron motion.

F. Energy-momentum tensor.

The operators $\underline{p}_\mu$ or, equivalently, $\underline{p}^\mu = \gamma^\mu \cdot \gamma^\nu \underline{p}_\nu$ acquire a physical meaning when used to define the components $T^{\mu\nu}$ of the electron *energy-momentum tensor:*

$$T^{\mu\nu} = T^\mu \cdot \gamma^\nu = (\gamma_0 \widetilde{\psi} \gamma^\mu \underline{p}^\nu \psi)_{(0)} = (\psi^\dagger \gamma_0 \gamma^\mu \underline{p}^\nu \psi)_{(0)} \,. \tag{226}$$

Its matrix equivalent is given in Table II. As mentioned in the discussion of the electromagnetic energy-momentum tensor,

$$T^\mu = T(\gamma^\mu) = T^{\mu\nu} \gamma_\nu \tag{227}$$

is the energy-momentum flux through a hyperplane with normal γ^μ. The energy-momentum density in the electron rest system is

$$T(v) = v_\mu T^\mu = \rho \, p \,. \tag{228}$$

This defines the *"expected" proper momentum* $p = p(x)$. The observable $p = p(x)$ is a statistical prediction for the momentum of the electron at x. In general, the momentum p is not collinear with the velocity, because it includes a contribution from the spin. A measure of this noncollinearity is $p \wedge v$, which should be recognized as defining the *relative* momentum in the electron rest frame.

From the definition (226) of $T^{\mu\nu}$ in terms of the Dirac wave function, momentum and angular momentum conservation laws can be established by direct calculation from the Dirac equation. First, it is found that[17]

$$\partial_\mu T^\mu = F \cdot J \,, \tag{229}$$

where $F = \nabla \wedge A$ is the electromagnetic field and

$$J = e\psi\gamma_0\widetilde{\psi} = e\rho v \tag{230}$$

is identified as the *charge current* (density), so charge conservation $\nabla \cdot J = 0$ is an immediate consequence of probability conservation. The right side of (229) is *exactly* the classical Lorentz force, so using (169) and denoting the electromagnetic energy-momentum tensor (168) by T^μ_{EM}, we can rephrase (229) as *the total energy-momentum conservation law*

$$\partial_\mu(T^\mu + T^\mu_{EM}) = 0 \,. \tag{231}$$

This justifies identifying the Dirac current with the charge current of the electron.

G. Angular momentum conservation.

To derive the angular momentum conservation law, we identify $T^\mu \wedge x$ as the orbital angular momentum tensor (See Table II for comparison with more conventional expressions). Noting that $\partial_\mu x = \gamma_\mu$, we calculate

$$\partial_\mu(T^\mu \wedge x) = T^\mu \wedge \gamma_\mu + \partial_\mu T^\mu \wedge x \,. \tag{232}$$

To evaluate the first term on the right, we return to the definition (226) and find

$$\begin{aligned} \gamma_\mu T^{\mu\nu} &= [\,(\underline{p}^\nu\psi)\gamma_0\widetilde{\psi}\,]_{(1)} = \tfrac{1}{2}\big[\,(\underline{p}^\nu\psi)\gamma_0\widetilde{\psi} + \psi\gamma_0(\underline{p}^\nu\psi)\tilde{}\,\big] \\ &= (\underline{p}^\nu\psi)\gamma_0\widetilde{\psi} - \partial^\nu(\tfrac{1}{2}\hbar\psi i\gamma_3\widetilde{\psi}) \,. \end{aligned} \tag{233}$$

Summing with γ_ν and using the Dirac equation (225) to evaluate the first term on the right while recognizing the spin vector (211) in the second term, we obtain

$$\gamma_\nu\gamma_\mu T^{\mu\nu} = m\psi\widetilde{\psi} + \nabla(\rho s i) \,. \tag{234}$$

The scalar part gives the curious result

$$T^\mu{}_\mu = T^\mu \cdot \gamma_\mu = m\rho\cos\beta \,. \tag{235}$$

However, the bivector part gives the relation we are looking for:

$$T^\mu \wedge \gamma_\mu = T^{\nu\mu}\gamma_\nu \wedge \gamma_\mu = \nabla \cdot (\rho s i) = -\partial_\mu(\rho S^\mu) \,, \tag{236}$$

where

$$S^\mu = (is)\cdot\gamma^\mu = i(s\wedge\gamma^\mu) \tag{237}$$

is the spin angular momentum tensor already identified in (213) and (217). Thus from (232) and (229) we obtain the angular momentum conservation law

$$\partial_\mu J^\mu = (F\cdot J)\wedge x\,, \tag{238}$$

where

$$J(\gamma^\mu) = J^\mu = T^\mu\wedge x + \rho S^\mu \tag{239}$$

is the bivector-valued angular momentum tensor, representing the total angular momentum flux in the γ^μ direction. In the electron rest system, therefore, the angular momentum density is

$$J(v) = \rho(p\wedge x + S)\,, \tag{240}$$

where recalling (199), $p\wedge x$ is recognized as the expected orbital angular momentum and as already advertised in (212), $S = isv$ can be identified as an intrinsic angular momentum or spin. This completes the justification for interpreting S as spin. The task remaining is to dig deeper and understand its origin.

H. Local observables.

We now have a complete set of conservation laws for the *local observables* $\rho,\ v,\ S$ and p, but we still need to ascertain precisely how the *kinetic momentum* p is related to the wave function. For that purpose we employ the invariant decomposition $\psi = (\rho e^{i\beta})^{\frac{1}{2}}R$. First we need some kinematics. By differentiating $R\widetilde{R} = 1$, it is easy to prove that the derivatives of the rotor R must have the form

$$\partial_\mu R = \tfrac{1}{2}\Omega_\mu R\,, \tag{241}$$

where $\Omega_\mu = \Omega_\mu(x)$ is a bivector field. Consequently the derivatives of the e_ν defined by (205) have the form

$$\partial_\mu e_\nu = \Omega_\mu\cdot e_\nu\,. \tag{242}$$

Thus Ω_μ is the *rotation rate* of the frame $\{e_\nu\}$ as it is displaced in the direction γ_μ.

Now, with the help of (212), the effect of $\underline{p}_\nu$ on ψ can be put in the form

$$\underline{p}_\nu\psi = [\,\partial_\nu(\ln\rho + i\beta) + \Omega_\nu\,]S\psi - eA_\nu\psi\,, \tag{243}$$

whence

$$(\underline{p}_\nu\psi)\gamma_0\widetilde{\psi} = [\,\partial_\nu(\ln\rho + i\beta) + \Omega_\nu\,]i\rho s - eA_\nu\rho v\,. \tag{244}$$

Inserting this in the definition (226) for the energy-momentum tensor, after some manipulations beginning with $is = Sv$, we get the explicit expression

$$T_{\mu\nu} = \rho\big[\,v_\mu(\Omega_\nu\cdot S - eA_\nu) - (\gamma_\mu\wedge v)\cdot(\partial_\nu S) - s_\mu\partial_\nu\beta\,\big]\,. \tag{245}$$

From this we find, by (228), the momentum components

$$p_\nu = \Omega_\nu \cdot S - eA_\nu \,. \tag{246}$$

This reveals that (apart from the A_ν contribution) the *momentum has a kinematical meaning related to the spin:* It is completely determined by the component of Ω_ν in the spin plane. In other words, it describes the *rotation rate* of the frame $\{e_\mu\}$ in the spin plane or, if you will "about the spin axis." But we have identified the angle of rotation in this plane with the phase of the wave function. Thus, the momentum describes the phase change in all directions of the wave function or, equivalently, of the frame $\{e_\mu\}$. A physical interpretation for this geometrical fact will be offered in the next Section.

The kinematical import of the operator $\underline{p}_\nu$ is derived from its action on the rotor R. To make that explicit, write (241) in the form

$$(\partial_\nu R)\mathbf{i}\hbar\widetilde{R} = \Omega_\nu S = \Omega_\nu \cdot S + \Omega_\nu \wedge S + \partial_\nu S \,, \tag{247}$$

where (212) was used to establish that

$$\partial_\nu S = \tfrac{1}{2}[\,\Omega_\nu, S\,] = \tfrac{1}{2}(\Omega_\nu S - S\Omega_\nu) \,. \tag{248}$$

Introducing the abbreviation

$$iq_\nu = \Omega_\nu \wedge S \,, \tag{249}$$

and using (246) we can put (247) in the form

$$(\underline{p}_\nu R)\widetilde{R} = p_\nu + iq_\nu + \partial_\nu S \,. \tag{250}$$

This shows explicitly how the operator $\underline{p}_\nu$ relates to kinematical observables, although the physical significance of q_ν is obscure. Note that both p_ν and $\partial_\nu S$ contribute to $T_{\mu\nu}$ in (245), but q_ν does not. By the way, it should be noted that the last two terms in (245) describe energy-momentum flux orthogonal to the v direction. It is altogether natural that this flux should depend on the component of $\partial_\nu S$ as shown. However, the significance of the parameter β in the last term of (245) remains obscure.

An auxiliary conservation law can be derived from the Dirac equation by decomposing the Dirac current as follows. Solving (225) for the Dirac charge current, we have

$$J = e\psi\gamma_0\widetilde{\psi} = \frac{e}{m}\,\gamma^\mu(\underline{p}_\mu\psi)\widetilde{\psi} \,. \tag{251}$$

The identity (250) is easily generalized to

$$(\underline{p}_\mu\psi)\widetilde{\psi} = (p_\mu + iq_\mu)\rho e^{i\beta} + \partial_\mu(\rho S e^{i\beta}) \,. \tag{252}$$

The right side exhibits the scalar, pseudoscalar and bivector parts explicitly. From the scalar part we define the *Gordon current:*

$$K_\mu = \frac{e}{m}\,[\,(\underline{p}_\mu\psi)\widetilde{\psi}\,]_{(0)} = \frac{e}{m}\,(\widetilde{\psi}\underline{p}_\mu\psi)_{(0)} = \frac{e}{m}\,(p_\mu\rho\cos\beta - q_\mu\rho\sin\beta) \,. \tag{253}$$

Or in vector form,

$$K = \frac{e}{m}\,\rho(p\cos\beta - q\sin\beta)\,. \tag{254}$$

When (252) is inserted into (251), the pseudovector part must vanish, and the vector part gives us the so-called *"Gordon decomposition"*

$$J = K + \nabla\cdot M, \tag{255}$$

where the definition (219) of the magnetization tensor M has been introduced for the last term in (252) This is ostensibly a decomposition into a *conduction current* K and a *magnetization current* $\nabla\cdot M$, both of which are separately conserved. But how does this square with the physical interpretation already ascribed to J? The possibility that it arises from a substructure in the charge flow is considered in the next Section.

So far we have supplied a physical interpretation for all parameters in the wave function (204) except the *"duality parameter"* β. To date, this parameter has defied all efforts at physical interpretation, because of its peculiar "duality role." For example, a straightforward interpretation of the Gordon current in (254) as a conduction current is confounded by $\beta \neq 0$. Similarly, equation (219) tells us that the magnetization (magnetic moment density) M is not directly proportional to the spin (as commonly supposed) but "dually proportional." The duality factor $e^{i\beta}$ has the effect of generating an effective electric dipole moment for the electron, as is easily shown by applying the spacetime split (181) to M. This seems to conflict with experimental evidence that the electron has no detectable electric moment, but the issue is subtle. We are forced to leave the problem of interpreting β as unresolved, though it rises again in the next Section.

VIII. Interpretation of Quantum Mechanics

Quantum mechanics has been spectacularly successful over an immense range of applications, so there is little doubt about the efficacy of its mathematical formulation. However, the physical interpretation of quantum mechanics has remained a matter of intense debate. Two prominent alternatives have emerged in the literature: the *Copenhagen interpretation* championed by Niels Bohr, and the *causal interpretation* championed by David Bohm. These two interpretations are so radically different as to constitute *different physical theories*, though they share the same mathematical formulation. The essential difference is that the causal theory asserts that *electrons have continuous paths in spacetime*, whereas the Copenhagen theory denies that.

James Cushing[21] has traced the history of the dispute between these theories and critically reviewed arguments in support of the causal theory. In agreement with many other commentators, he concludes that the causal theory is perfectly viable, and every objection from the Copenhagen camp has been adequately answered in the literature. He traces the dispute from the inception of quantum

mechanics and comes to the surprising conclusion that the dominance of the Copenhagen theory in the physics literature is a historical accident that could easily have been deflected in favor of the causal theory instead.

Our real formulation of Schroedinger-Pauli-Dirac theory puts the causal-Copenhagen dispute in new light by making the geometric structure of the equations more explicit. The causal theory admits to a much more detailed physical interpretation of this structure than the Copenhagen theory, including hidden structure revealed by the real formulation. However, since real QM is mathematically isomorphic with standard QM, our analysis does not contradict successes of the Copenhagen theory.

Real QM does raise some questions for Copenhagen theory though. First, questions about the relation of observables to operators in QM are raised by the realization that both hermiticity and noncommutivity of Pauli and Dirac matrices have clear geometric meanings with no necessary connection to QM. Second, any interpretation of uncertainty relations should account for the fact that Planck's constant enters the Dirac equation only as the magnitude of the spin. What indeed does spin have to do with limitations on observability?

The causal theory does not resolve all the mysteries of QM. Rather it replaces the mysteries of Copenhagen theory with a different set of mysteries. As the two theories are mathematically equivalent, the choice between them could be regarded as a matter of taste. However, they suggest very different directions for research that could lead to testable differences between them.

Our discussion here is concentrated on the geometry of the single particle Dirac theory as a guide to physical interpretation. Many particle theory raises new issues. We merely note that Bohm and his followers have extended the causal theory to the many particle case[22,23] and demonstrated its use in explaining such mysterious QM effects as entanglement. As real QM is so similar to Bohm's theory in the one-particle case, it has a straightforward extension to the many-particle case by following Bohm. No position on the validity of that extension is taken here.

A. Electron trajectories.

In classical theory the concept of *particle* refers to an object of negligible size with a continuous trajectory. Copenhagen theory asserts that it is meaningless or impossible in quantum mechanics to regard the electron as a particle in this sense. On the contrary, Bohm argues that the difference between classical and quantum mechanics is not in the concept of particle itself but in the equation for particle trajectories. From Schroedinger's equation he derived an equation of motion for the electron that differs from the classical equation only in a statistical term called the "quantum force." He was careful, however, not to commit himself to any special hypothesis about the origins of the quantum force. He accepted the form of the force dictated by Schroedinger's equation, and he took pains to show that all implications of Schroedinger theory are compatible with a strict particle interpretation. Adopting the same general particle interpretation of the Dirac theory, we find a generalization of Bohm's equation that provides a new perspective on the quantum force.

The Dirac current ρv assigns a unit timelike vector $v(x)$ to each spacetime

point x where $\rho \neq 0$. In accordance with the causal theory, we interpret $v(x)$ as the *expected* proper *velocity* of the electron at x, that is, the velocity predicted for the electron if it happens to be at x. The velocity $v(x)$ defines a local reference frame at x called the *electron rest frame*. The *proper probability density* $\rho = (\rho v) \cdot v$ can be interpreted as the probability density in the rest frame. By a well known theorem, the probability conservation law (208) implies that through each spacetime point there passes a *unique integral curve* that is tangent to v at each of its points. Let us call these curves (*electron*) *streamlines*. In any spacetime region where $\rho \neq 0$, a solution of the Dirac equation determines a *family of streamlines* that fills the region with exactly one streamline through each point. The streamline through a specific point x_0 is the expected history of an electron at x_0, that is, it is the optimal prediction for the history of an electron that actually is at x_0 (with relative probability $\rho(x_0)$, of course!). Parametrized by proper time τ, the streamline $x = x(\tau)$ is determined by the equation

$$\frac{dx}{d\tau} = v(x(\tau))\,. \tag{256}$$

The main objection to a strict particle interpretation of the Schroedinger and Dirac theories is the Copenhagen claim that a wave interpretation is essential to explain diffraction. The causal theory claims otherwise, based on the *fact* that the wave function determines a unique family of electron trajectories. For double slit diffraction these trajectories have been calculated from Schroedinger's equation,[24,25] and, recently, from the Dirac equation.[15] Sure enough, after flowing uniformly through the slits, the trajectories bunch up at diffraction maxima and thin out at the minima. According to Bohm, the cause of this phenomenon is the quantum force rather than wave interference. This shows at least that the particle interpretation is not inconsistent with diffraction phenomena, though the origin of the quantum force remains to be explained. The obvious objections to this account of diffraction have been adequately refuted in the literature.[21] It is worth noting, though, that this account has the decided advantage of avoiding the paradoxical "collapse of the wave function" inherent in the "dualist" Copenhagen explanation of diffraction. At no time is it claimed that the electron spreads out like a wave to interfere with itself and then "collapses" when it is detected in a localized region. The claim is only that the electron is likely to travel on one of a family of possible trajectories consistent with experimental constraints; which trajectory is known initially only with a certain probability, though it can be inferred more precisely after detection in the final state. Indeed, it is possible then to infer which slit the electron passed through.[24] These remarks apply to the Dirac theory as well as to the Schroedinger theory, though there are some differences in the predicted trajectories,[15] because the Schroedinger current is the nonrelativistic limit of the Gordon current rather than the Dirac current.[29]

Now let us investigate the equations for motion along a Dirac streamline $x = x(\tau)$. On this curve the kinematical factor in the Dirac wave function (204)

can be expressed as a function of proper time

$$R = R(x(\tau))\,. \tag{257}$$

By (205), (207) and (256), this determines a *comoving frame*

$$e_\mu = R\,\gamma_\mu \widetilde{R} \tag{258}$$

on the streamline with velocity $v = e_0$, while the spin vector s and bivector S are given as before by (211) and (212). In accordance with (241), differentiation of (257) leads to

$$\dot{R} = v \cdot \nabla R = \tfrac{1}{2}\Omega R\,, \tag{259}$$

where the overdot indicates differentiation with respect to proper time, and

$$\Omega = v^\mu \Omega_\mu = \Omega(x(\tau)) \tag{260}$$

is the rotational velocity of the frame $\{e_\mu\}$. Accordingly,

$$\dot{e}_\mu = v \cdot \nabla\, e_\mu = \Omega \cdot e_\mu\,. \tag{261}$$

But these equations are identical in form to those in Section V for the classical theory of a relativistic rigid body with negligible size. This is a consequence of the particle interpretation. In Bohmian terms, the only difference between classical and quantum theory is in the functional form of Ω. Our main task, therefore, is to investigate what the Dirac theory tells us about Ω.

We begin by examining the special case of a free particle and the simplest approach to the classical limit. Then we formulate the causal theory in the most general terms and discuss its extension to a more detailed interpretation of Dirac theory.

B. Solutions of the Dirac equation.

This is not the place for a systematic study of solutions to the Dirac equation. Suffice it to say that every solution in the matrix theory has a corresponding solution in the real theory. To show what a "real solution" looks like and the physical insight that it offers, we consider the simplest example of a free particle.

For a free particle with proper momentum p, the wave function ψ is an eigenstate of the "proper momentum operator" (223), that is,

$$\underline{p}\psi = p\psi, \tag{262}$$

so the Dirac equation (225) reduces to the algebraic equation

$$p\psi = \psi\gamma_0 m. \tag{263}$$

The solution is a *plane wave* of the form

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}} R = \rho^{\frac{1}{2}} e^{i\beta/2} R_0 e^{-\mathbf{i}p\cdot x/\hbar}\,, \tag{264}$$

where the kinematical factor R has been decomposed to explicitly exhibit its spacetime dependence on a phase satisfying $\nabla p \cdot x = p$. Inserting this into (223) and solving for p we get

$$p = me^{i\beta} R\gamma_0 \widetilde{R} = mve^{-i\beta}\,. \tag{265}$$

This implies $e^{i\beta} = \pm 1$, so

$$e^{i\beta/2} = 1 \text{ or } i\,, \tag{266}$$

and $p = \pm mv$ corresponding to two distinct solutions. One solution appears to have negative energy $E = p \cdot \gamma_0$, but that can be rectified by changing the sign in the phase of the "trial solution" (264).

Thus we obtain two distinct kinds of plane wave solutions with positive energy $E = p \cdot \gamma_0$:

$$\psi_- = \rho^{\frac{1}{2}} R_0 e^{-\mathbf{i}p\cdot x/\hbar}\,, \tag{267}$$

$$\psi_+ = \rho^{\frac{1}{2}} i R_0 e^{+\mathbf{i}p\cdot x/\hbar}\,. \tag{268}$$

We identify these as *electron* and *positron* wave functions. Indeed, the two solutions are related by charge conjugation. According to (202), the charge conjugate of (267) is

$$\psi_-^C = \psi_- \boldsymbol{\sigma}_2 = \rho^{\frac{1}{2}} i R_0' e^{+\mathbf{i}p\cdot x/\hbar}\,, \tag{269}$$

where

$$R_0' = R_0(-i\boldsymbol{\sigma}_2)\,. \tag{270}$$

The factor $-i\boldsymbol{\sigma}_2$ represents a spatial rotation that just "flips" the direction of the spin vector. Evidently (268) and (269) are both positron solutions, but with oppositely directed spins.

Determining the comoving frame (258) for the electron solution (267), we find that the velocity $v = R_0\gamma_0\widetilde{R}_0$ and the spin $s = \frac{1}{2}\hbar R_0\gamma_3\widetilde{R}_0$ are constant, but, for $k = 1, 2$,

$$e_k(\tau) = e_k(0)e^{-p\cdot x/S} = e_k(0)e^{e_2 e_1 \omega\tau}\,, \tag{271}$$

where $\tau = v \cdot x$ is the proper time along the streamline and frequency ω is given by

$$\omega = \frac{2m}{\hbar} = 1.6 \times 10^{21}\,\mathrm{s}^{-1}\,. \tag{272}$$

Thus, the streamlines are straight lines along which the spin is constant, and e_1 and e_2 rotate about the "spin axis" with the ultrahigh frequency (272) as the electron moves along the streamline. A similar result is found for the positron solution.

For applications, the constants in the solution must be specified in more detail. If the wave functions are normalized to one particle per unit volume V in the γ_0-system, then we have

$$\rho_0 = \gamma_0 \cdot (\rho v) = \frac{1}{V} \quad \text{or} \quad \rho = \frac{m}{EV} = \frac{1}{\gamma_0 \cdot vV}\,. \tag{273}$$

To separate velocity and spin variables, we follow the procedure beginning with (80) to make the spacetime split

$$R = LU \qquad \text{where} \qquad U = U_0 e^{-\mathbf{i}p\cdot x/\hbar}\,. \tag{274}$$

Inserting this into (263), we can express L in terms of p and γ_0, as already shown in (88). The rotor U describes the spin direction in the same way as in the Pauli theory in GA1.

C. The classical limit.

One way to get a classical limit is through an "eikonal approximation" to the Dirac equation. Accordingly, the wave function is set in the form

$$\psi = \psi_0 e^{-\mathbf{i}\varphi/\hbar}\,. \tag{275}$$

Then the "amplitude" ψ_0 is assumed to be slowly varying compared to the "phase" φ, so the derivatives of ψ_0 in the Dirac equation can be neglected to a good approximation. Thus, inserting (275) into the Dirac equation, say in the form (192), we obtain

$$(\nabla\varphi - eA)e^{i\beta} = mv\,. \tag{276}$$

As in the plane wave case (265) this implies $e^{i\beta} = \pm 1$, and the two values correspond to electron and positron solutions. For the electron case,

$$\nabla\varphi - eA = mv\,. \tag{277}$$

This defines a family of classical histories in spacetime. For a given external potential $A = A(x)$, the phase φ can be found by solving the "Hamilton-Jacobi equation"

$$(\nabla\varphi - eA)^2 = m^2\,, \tag{278}$$

obtained by squaring (277). On the other hand, the curl of (277) gives

$$m\nabla \wedge v = -e\nabla \wedge A = -eF\,. \tag{279}$$

Dotting this with v and using the identity

$$\dot{v} = v\cdot(\nabla \wedge v) = v\cdot\nabla v, \tag{280}$$

we obtain exactly the classical Lorentz force for each streamline.

Inserting (279) into (287), we obtain

$$\Omega = \frac{e}{m}\,F + (m + eA\cdot v)S^{-1}\,. \tag{281}$$

Whence the rotor equation (259) assumes the explicit form

$$\dot{R} = \frac{e}{2m}\,FR - R\mathbf{i}(m + eA\cdot v)/\hbar\,. \tag{282}$$

This admits a solution by separation of variables:

$$R = R_0 e^{-\mathbf{i}\varphi/\hbar}\,, \tag{283}$$

where

$$\dot{R}_0 = \frac{e}{2m} F R_0 \tag{284}$$

and

$$\dot{\varphi} = v \cdot \nabla \varphi = m + eA \cdot v \,. \tag{285}$$

Equation (284) is identical with the classical rotor equation (99) with *Lorentz torque*, while (285) can be obtained from (277).

Thus, in the eikonal approximation the quantum equation for a comoving frame differs from the classical equation only in having additional rotation in the spin plane. Quantum mechanics also assigns energy to this rotation, and an explicit expression for it is obtained by inserting (281) into (246), with the interesting result

$$p \cdot v = m + \frac{e}{m} F \cdot S \,. \tag{286}$$

This is what one would expect classically if there were some sort of localized motion in the spin plane. Note that the high frequency rotation rate (272) due to the mass is shifted by a magnetic type interaction. That possibility is considered below.

The two kinds of solutions distinguished by the values of β in (266) and (276) suggest that β parametrizes an admixture of particle–antiparticle states. Unfortunately, that is inconsistent with more general solutions of the Dirac equation, such as the Darwin solutions for the Hydrogen atom. One way out of the dilemma is simply to assert that it shows the need for second quantization, but that solution is too facile without further argument.

D. Quantum torque

Having gained some physical insight from special cases, let us turn to the derivation of a general equation for a Dirac streamline. For this purpose, we know that the rotor equation (259) is optimal. All we need is an explicit form for the rotational velocity Ω defined by (260). A general expression for Ω in terms of observables has been derived from the Dirac equation in two steps.[17] The first step yields the interesting result

$$\Omega = -\nabla \wedge v + v \cdot (i \nabla \beta) + (m \cos\beta + eA \cdot v) S^{-1} \,. \tag{287}$$

But this tells us nothing about particle streamlines, because it gives us the identity (280) for the velocity. The second step yields

$$-\nabla \wedge v + v \cdot (i \nabla \beta) = m^{-1} (eF e^{i\beta} + Q) \,, \tag{288}$$

where Q has the complicated form

$$Q = -e^{i\beta} \{ \partial_\mu W^\mu + \frac{1}{2} (\gamma_\mu \wedge \gamma_\nu) [(W^\mu \times W^\nu) S^{-1}] \}_{(2)} \,, \tag{289}$$

where $A \times B$ is the commutator product and

$$W_\mu = (\rho e^{i\beta})^{-1} \partial_\mu (\rho e^{i\beta} S) = \partial_\mu S + S \partial_\mu (\ln \rho + i\beta) \,. \tag{290}$$

Hence,

$$\Omega = \frac{e}{m} F e^{i\beta} + m^{-1} Q + (m \cos\beta + eA \cdot v) S^{-1} \,. \tag{291}$$

This is the desired result in its most general form.

Again we see the Lorentz torque in (291), but multiplied by the duality factor $e^{i\beta}$. Again the cases with opposite charge are covered by $\cos\beta = \pm 1$, and that assignment simplifies the other terms in (291) as well. However, the value of β is set by solving the Dirac equation, and in solutions for the Hydrogen atom, for example, β is a variable function of position that so far has defied physical interpretation.

The term Q in (291) generalizes the "quantum force" term that Bohm identifies in Schroedinger theory as responsible for quantum effects on particle motion.[22] Like the "Lorentz torque" it exerts a torque on the spin as well as a force on the motion, so let us call Q the *quantum torque.* From (290) we see that Q is independent of normalization on the probability density ρ, as Bohm has observed for the quantum force. However, the striking new insight brought by the Dirac theory and made explicit by (289) and (290) is that the quantum torque is derived from spin. To put it baldly: No spin! No quantum torque! No quantum force! No quantum effects! This may be the strongest theoretical evidence that *spin is an essential ingredient of QM*, not simply an "add-on" to more basic quantum behavior.

Though Bohm never noticed it, the quantum force is spin dependent even in Schroedinger theory, provided it is derived from Dirac theory.[29]

The expression (291) for Ω may be the best starting place for studying the classical limit. The *classical limit* can be characterized first by $\partial_\mu \ln\rho \to 0$ and, say, $\cos\beta = 1$; second, by $\partial_\mu S = v_\mu \dot{S}$, which comes from assuming that only the variation of S along the history can affect the motion. Accordingly, (289) reduces to $Q = \ddot{S}$, and for the limiting classical equations of motion for a particle with intrinsic spin we obtain

$$m\dot{v} = (eF - \ddot{S}) \cdot v \,, \tag{292}$$

$$m\dot{S} = (eF - \ddot{S}) \times S \,. \tag{293}$$

These coupled equations have not been seriously studied. Of course, they should be studied in conjunction with the spinor equation (259).

E. Zitterbewegung.

Many students of the Dirac theory including Schroedinger[26,27] and Bohm[22] have suggested that the spin of a Dirac electron is generated by localized particle circulation that Schroedinger called *zitterbewegung* (= trembling motion). Schroedinger's original analysis applied only to free particles. However, the real Dirac theory provides a natural extension of the interpretation to all solutions of the Dirac equation. Since the Dirac equation is the prototypical equation for all fermions, the interpretation extends broadly to quantum mechanics. It has been dubbed the *the zitterbewegung (zbw) interpretation of quantum mechanics.*[28]

The zbw interpretation can be regarded as a refinement of the causal interpretation of QM, so it needs to be evaluated in the same light. Its main advantage is the simple, coherent picture it gives for electron motion. Here is a brief introduction to the idea.

We have seen that the kinematics of electron motion is completely characterized by the "Dirac rotor" R in the invariant decomposition (204) of the wave function. The Dirac rotor determines a comoving frame $\{e_\mu = R\gamma_\mu\widetilde{R}\}$ that rotates at high frequency in the e_2e_1-plane, the "spin plane," as the electron moves along a streamline. Moreover, according to (286) there is energy associated with this rotation, indeed, all the rest energy $p \cdot v$ of the electron. These facts suggest that the *electron mass, spin and magnetic moment are manifestations of a local circular motion of the electron.* Mindful that the velocity attributed to the electron is an independent assumption imposed on the Dirac theory from physical considerations, we recognize that this suggestion can be accommodated by giving the electron a component of velocity in the spin plane. Accordingly, we now *define the electron velocity u by*

$$u = v - e_2 = e_0 - e_2 \,. \tag{294}$$

The choice $u^2 = 0$ has the advantage that the electron mass can be attributed to kinetic energy of self interaction while the spin is the corresponding angular momentum.[28]

This new identification of electron velocity makes the plane wave solutions more physically meaningful. For $p \cdot x = mv \cdot x = m\tau$, the kinematical factor for the solution (267) can be written in the form

$$R = e^{\frac{1}{2}\Omega\tau}R_0 \,, \tag{295}$$

where Ω is the constant bivector

$$\Omega = mS^{-1} = \frac{2m}{\hbar}\, e_1e_2 \,. \tag{296}$$

From (295) it follows that v is constant and

$$e_2(\tau) = e^{\Omega\tau}e_2(0) \,. \tag{297}$$

So $u = \dot{z}$ can be integrated immediately to get the electron history

$$z(\tau) = v\tau + (e^{\Omega\tau} - 1)r_0 + z_0 \,, \tag{298}$$

where $r_0 = \Omega^{-1}e_2(0)$. This is a lightlike helix centered on the Dirac streamline $x(\tau) = v\tau + z_0 - r_0$. In the electron "rest system" defined by v, it projects to a circular orbit of radius

$$|\, r_0 \,| = |\, \Omega^{-1} \,| = \frac{\hbar}{2m} = 1.9 \times 10^{-13}\mathrm{m} \,. \tag{299}$$

The diameter of the orbit is thus equal to an electron Compton wavelength. For $r(\tau) = e^{\Omega\tau}r_0$, the angular momentum of this circular motion is, as intended, the spin

$$(m\dot{r}) \wedge r = m\dot{r}r = mr^2\Omega = m\Omega^{-1} = S \,. \tag{300}$$

Finally, if z_0 is varied parametrically over a hyperplane normal to v, equation (298) describes a 3-parameter family of spacetime filling lightlike helixes, each centered on a unique Dirac streamline. According to the causal interpretation, the electron can be on any one of these helixes with uniform probability.

Let us refer to this localized helical motion of the electron by the name *zitterbewegung* (*zbw*) originally introduced by Schroedinger. Accordingly, we call $\omega = \Omega \cdot S$ the *zbw frequency* and $\lambda = \omega^{-1}$ the *zbw radius*. The phase of the wave function can now be interpreted literally as the phase in the circular motion, so we can refer to that as the *zbw phase*.

Although the frequency and radius ascribed to the *zbw* are the same here as in Schroedinger's work, its role in the theory is quite different. Schroedinger attributed it to interference between positive and negative energy components of a wave packet,[26,27] whereas here it is associated directly with the complex phase factor of a plane wave. From the present point of view, wave packets and interference are not essential ingredients of the *zbw*, although the phenomenon noticed by Schroedinger certainly appears when wave packets are constructed. Of course, the present interpretation was not an option open to Schroedinger, because the association of the unit imaginary with spin was not established (or even dreamed of), and the vector e_2 needed to form the spacelike component of the *zbw velocity* u was buried out of sight in the matrix formalism. Now that it has been exhumed, we can see that the *zbw* may play a ubiquitous role in quantum mechanics. The present approach associates the *zbw* phase and frequency with the phase and frequency of the complex phase factor in the electron wave function. This is the central feature of the *the zitterbewegung interpretation* of quantum mechanics.

The strength of the *zbw* interpretation lies first in its coherence and completeness in the Dirac theory and second in the intimations it gives of more fundamental physics. It will be noted that the *zbw* interpretation is completely general, because the definition (294) of the *zbw* velocity is well defined for any solution of the Dirac equation. It is also perfectly compatible with everything said about the causal interpretation of the Dirac theory. One need only recognize that the Dirac velocity can be interpreted as the average of the electron velocity over a *zbw* period, as expressed by writing

$$v = \bar{u} \,. \tag{301}$$

Since the period is on the order of 10^{-21}s, it is v rather than u that best describes electron motion in most experiments.

Perhaps the strongest theoretical support for the *zbw* interpretation is the fact that it is fundamentally geometrical; it completes the kinematical interpretation of R, so all components of R, even the complex phase factor, characterize features of the electron history.

The *key ingredients* of the *zbw* interpretation are the complex phase factor and the energy-momentum operators $\underline{p}_\mu$ defined by (223). The unit imaginary $\mathbf{i}$ appearing in both of these has the dual properties of representing the plane in which *zbw* circulation takes place and generating rotations in that plane. The phase factor literally represents a rotation on the electron's circular orbit in the

spin plane. Operating on the phase factor, the $\underset{\_}{p}_\mu$ computes the phase rotation rates in all spacetime directions and associates them with the electron energy-momentum. Thus, the *zbw* interpretation explains the physical significance of the mysterious "quantum mechanical operators" $\underset{\_}{p}_\mu$.

The key ingredients of the *zbw* interpretation are preserved in the nonrelativistic limit and so provide a *zitterbewegung interpretation of Pauli-Schroedinger theory.* The nonrelativistic approximation to the STA version of the Dirac theory, leading through the Pauli theory to the Schroedinger theory, has been treated in detail elsewhere.[18] But the essential point can be seen by a split of the Dirac wave function ψ into the factors

$$\psi = \rho^{\frac{1}{2}} e^{i\beta/2} L U e^{-\mathbf{i}(m/\hbar)t} \,. \tag{302}$$

In the nonrelativistic approximation three of these factors are neglected or eliminated and ψ is reduced to the Pauli wave function $\psi_P = \rho^{\frac{1}{2}} U$, where the rotor U retains the portion of the phase that is influenced by external interactions.

It follows that even in the Schroedinger theory the phase $\varphi/\hbar$ describes the *zbw*, and $\partial_\mu \varphi$ describes the *zbw* energy and momentum. This implies that the physical significance of the complex phase factor $e^{-\mathbf{i}(\varphi/\hbar)}$ is kinematical rather than logical or statistical as so often claimed.

The *zbw* interpretation has the potential to explain much more than the electron spin and magnetic moment,[28,30,31] but it remains to be seen if that is a fruitful direction for research.

One interesting direction for future research is application of Feynman's path integral methods in real quantum theory. Suppose that the electron state at each point x is characterized by a spacetime rotor $R_k(x)$ for each path to the point. Feynman's complex phase factor can then be incorporated in the $R_k(x)$ as part of the zbw path, and spin will be included automatically. It is easy to prove that the sum over paths will then produce a wave function of the general form

$$\psi(x) = \sum_k R_k(x) = (\rho e^{i\beta})^{\frac{1}{2}} R \tag{303}$$

Thus, the factor$(\rho e^{i\beta})^{\frac{1}{2}}$ arises from superposition, which supports its interpretation as a statistical factor and may thereby explain the origin of the troublesome parameter β.

IX. STA in the physics curriculum

I claim that the physics curriculum at all levels can be thoroughly unified and considerably simplified by adopting STA as the core mathematical language of physics. The language is fully developed and ready to use. Setting the politics of curriculum reform aside, let us consider how a forward-looking physics department could incorporate STA into its curriculum.

In GA1 I made the case for adopting GA as the mathematical language of physics from the outset of the first course. For the sake of argument, let us suppose that has been done. Presumably, the students will have developed some proficiency with GA by the end of the first semester, or the first year, at least. That, I propose, is the ideal time to introduce the rudiments of STA, with the objective of developing student capacity for spacetime thinking as early as possible. This step is not so radical as might be supposed, for the fundamental geometric product defining STA in Section II is nearly the same as the defining product introduced in GA1 for classical physics, the main difference being the signature of spacetime and its geometric role in characterizing the light cone. Moreover, the spacetime split in Section III makes it possible to interrelate relativistic and nonrelativistic physics without appeal to Lorentz transformations. That enables students immediately to reason with relativistic invariants and acquire a working knowledge of such important physical cancepts as mass-energy equivalence, energy-momentum conservation and time dilation. This portion of the curriculum is easily constructed from available materials.[3]

Early introduction to STA will make it available throughout the rest of the curriculum, so it will be possible to move fluently between relativistic and nonrelativistic treatments of any topic, whatever is most appropriate. Wasted time in treating a topic both relativistically and nonrelativistically in different courses will be eliminated. The usual junior level course in electrodynamics will be able to take full advantage of the simplifications brought by the STA treatment in Sections V and VI. Finally, the senior level quantum mechanics course will be able to deal with the real Dirac equation from the outset. I daresay that this would be an eyeopener to many physicists.[15]

It should be recognized that this unprecedented simplification of classical, relativistic and quantum physics is enabled by two profound STA innovations: First, a common spinor method for rotations and rotational dynamics. Second, a universal concept of vector derivative.

Of course, the wholesale reconstruction of the physics curriculum proposed here will be a formidable task, though all the pieces are at hand. Who will volunteer to get it started?!

Note. Most of the papers listed in the references are available on line at <http://modelingnts.la.asu.edu> or <http://www.mrao.cam.ac.uk/~clifford/>.

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