Chapter 19
The Shape of Differential Geometry in Geometric Calculus

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Abstract We review the foundations for coordinate-free differential geometry in *Geometric Calculus*. In particular, we see how both extrinsic and intrinsic geometry of a manifold can be characterized a single bivector-valued one-form called the *Shape Operator*. The challenge is to adapt this formalism to *Conformal Geometric Algebra* for wide application in computer science and engineering.

19.1 Introduction

Geometric Algebra (GA) enabled the development of several new methods for *coordinate-free differential geometry* on manifolds of any dimension in [8]. In the most innovative of these methods, both extrinsic and intrinsic geometry of a manifold are characterized by a single bivector-valued one-form called the *shape operator*, which is essentially the derivative of the tangent space pseudoscalar as it slides along the manifold. I regard creation of this approach to differential geometry as some of my best work, so I am rather disappointed that, apart from one fine application [15], it has not been further exploited by me or anyone else.

As abundantly demonstrated in this book and elsewhere [2, 7], *Conformal Geometric Algebra* (CGA) has recently emerged as an ideal tool for computational geometry in computer science and engineering. My purpose here is to prepare the way for integrating the Shape Operator into the CGA tool kit for routine applications of differential geometry. I hope this will stimulate others to deal with practical implementation and applications.

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19.2 Geometric Calculus – basic concepts

Geometric Algebra is essential to formulate the basic concepts of "vector derivative" and "directed integral." Their initial formulations in [3] raised questions about relations to the Cartan's concept of "differential forms" [5]. That stimulated development of the *Geometric Calculus* (GC) in Chapters 4-7 of [8].

To elucidate the structure of Geometric Calculus, its basic concepts are listed here and their unique features are described in subsequent sections. The purpose is to explain how GC enables differential geometry without coordinates.

- **Universal Geometric Algebra** – arbitrary dimension and signature
- **Vector manifolds** – for representing any manifold
- **Directed integrals** and **differential forms**
- **Vector derivative** and the **fundamental theorem of calculus**
- **Differentials** and **codifferentials** for mappings and fields
- **Shape** and **curvature** for differential geometry

19.3 Differentiable Manifolds as Vector Manifolds

A (differentiable) manifold $\mathcal{M}^m$ of dimension m is a set on which differential and integral calculus is well-defined. The standard definition requires covering the manifold with overlapping charts of local coordinates. Calculus is then done indirectly by local mappings to $\mathbb{R}^m = \mathbb{R} \otimes \mathbb{R} \otimes ... \otimes \mathbb{R}$. Proofs are then required to establish that results are independent of coordinates.

In contrast, a *vector manifold* $\mathcal{M}^m = \{x\}$ of dimension m is defined as a set of vectors (called points) in GA that generates at each point x a *tangent space* with unit *pseudoscalar* $I_m(x)$. Any other manifold can then be defined as a set that is isomorphic to a vector manifold.

Thus, GC enables a concept of manifold that is manifestly coordinate-free. As we shall see, calculus can then be done directly with algebraic operations on points, and geometry is completely determined by derivatives of the pseudoscalar. It should be noted that a vector manifold can be defined without assuming that it is embedded in a vector space of specified dimension, though embedding theorems can no doubt be proved therefrom.

Though GC enables a coordinate-free approach to manifolds, it also provides a very efficient formalism for handling coordinates. That is worth reviewing briefly, because it facilitates direct connection to the standard literature and, of course, use of coordinates when appropriate.

The vector-valued function $x = x(x^1, x^2, ..., x^m)$ represents a patch of $\mathcal{M}^m$ parametrized by *scalar coordinates* (Fig. 19.1). The inverse mapping into $\mathbb{R}^m$ is given by *coordinate functions* $x^\mu = x^\mu(x)$. A *coordinate frame* $\{e_\mu = e_\mu(x)\}$

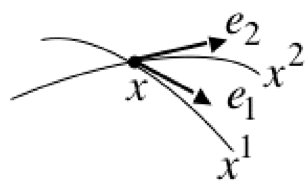

Fig. 19.1 Coordinate curves.

is defined by

$$e_\mu = \partial_\mu x = \frac{\partial x}{\partial x^\mu} = \lim \frac{\triangle x}{\triangle x^\mu}$$

with pseudoscalar

$$e_{(m)} = e_1 \wedge e_2 \wedge ... \wedge e_m = |e_{(m)}|\, I_m.$$

It is interesting to note that Elie Cartan used the expression $e_\mu = \partial_\mu x$ in an intuitive way at the foundation of his approach to differential geometry. Thus, GC provides the means to give it a more rigorous formulation.

Calculations with frames are greatly facilitated by employing a *reciprocal frame* $\{x^\mu\}$, often defined implicitly by the equations $e^\mu \cdot e_\nu = \delta^\mu_\nu$, which have the solution

$$e^\mu = (e_1 \wedge \dots \quad (\,)_\mu \wedge \dots \wedge e_m)\, e^{-1}_{(m)},$$

where the μth vector is omitted from the product. This can be used for a coordinate definition of the *vector derivative*, that is, the derivative with respect to the point x:

$$\partial = \partial_x = e^\mu \partial_\mu \quad \text{where} \quad \partial_\mu = e_\mu \cdot \partial = \frac{\partial}{\partial x^\mu}. \tag{19.1}$$

Consequently, the reciprocal vectors can be expressed as *gradients*:

$$e^\mu = \partial x^\mu.$$

The question remains: How can the vector derivative be defined without coordinates? The answer is given by first defining integration on vector manifolds, to which we now turn.

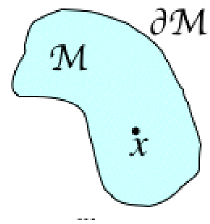

Fig. 19.2 Vector Manifold.

19.4 Directed Integrals and the Fundamental Theorem

Let $F = F(x)$ be a multivector-valued function on the manifold $\mathcal{M} = \mathcal{M}^m$ (Fig. 19.2) with a *directed measure* $d^m x = |d^m x| I_m(x)$. The measure can be expressed in terms coordinates by

$$d^m x = d_1 x \wedge d_2 x \wedge \ldots \wedge d_m x = e_1 \wedge e_2 \wedge \ldots \wedge e_m \; dx^1 dx^2 \ldots dx^m,$$

where $d_\mu x = e_\mu(x) d^\mu x$ (no sum). Accordingly, the usual scalar-valued volume element of integration is given by:

$$|d^m x| = |e_{(m)}| \, dx^1 dx^2 \ldots dx^m.$$

The directed integral of F can now be expressed as a standard multiple integral:

$$\int_{\mathcal{M}} d^m x F = \int_{\mathcal{M}} e_{(m)} \, dx^1 dx^2 \ldots dx^m.$$

This establishes contact with standard integration theory. It is worth mentioning that there are many practical and theoretical advantages to defining and evaluating the directed integral without reducing it to a multiple integral with scalar coordinates, though that cannot be addressed here.

Now we are equipped to formulate the *fundamental theorem of calculus* in the powerful general form that GC makes possible. We shall see that this leads us to a coordinate-free definition of the vector derivative in terms of the directed integral that reduces proof of the fundamental theorem to a near triviality. In addition, it generalizes the definition of derivative through (19.1) to apply to discontinuous functions (such as occur at the boundaries of material media in physics).

It is enlightening to begin with the important special case of a manifold embedded in a vector space: $\mathcal{M} = \mathcal{M}^m \subset \mathcal{V}^n$. Let $\nabla = \nabla_x$ denote the derivative of a point in the vector space $\mathcal{V}^n$. The derivative of any field $F = F(x)$ can then be decomposed algebraically into

$$\nabla F = \nabla \cdot F + \nabla \wedge F.$$

Thus GC unifies the familiar concepts of "*divergence*" and "*curl*" into a single vector derivative, which could well be dubbed the "*gradient*", as it reduces to the usual gradient when the field is scalar-valued.

Now we can formulate the first generalization of the fundamental theorem of calculus made possible by GC:

$$\int_{\mathcal{M}} (d^m x) \cdot \nabla F = \int_{\partial \mathcal{M}} d^{m-1} x F. \tag{19.2}$$

As explained in [3] when this was first written down, all the integral formulas of standard vector calculus (including those attributed to Gauss, Stokes and Green) are included as special cases of this formula.

I was puzzled for a while by the role of the inner product on the left side of (19.2). Then I realized its function is to project the derivative ∇ to a derivative on the submanifold $\mathcal{M}$, as expressed by

$$\partial = \partial_x = I_m^{-1}(I_m \cdot \nabla).$$

Hence, one can write $d^m x\, \partial = (d^m x) \cdot \partial = (d^m x) \cdot \nabla$, so the theorem (19.2) can be written in the form:

$$\int_{\mathcal{M}} d^m x\, \partial F = \int_{\partial \mathcal{M}} d^{m-1} x\, F, \tag{19.3}$$

which has no explicit reference to the embedding space. That observation inspired the following *coordinate-free definition for the vector derivative with respect to x in $\mathcal{M}$ without reference to any embedding space*:

$$\partial F = \lim_{d\omega \to 0} \frac{1}{d\omega} \oint d\sigma F, \tag{19.4}$$

where $d\omega = d^m x$ and $d\sigma = d^{m-1}x$. I called this the "*tangential derivative*," when I first proposed it, to emphasize that it is determined by the restriction of the variable x to $\mathcal{M}$. One consequence of that is that the operator $\partial = \partial_x$ is itself a function of x, so, for example, the theorem $\nabla \wedge \nabla = 0$, which holds for derivatives on a vector space (a "flat manifold"), does not apply for derivatives on a curved manifold, where $\partial \wedge \partial \neq 0$ in general. That property is essential for the formulation of differential geometry with GC, as we see below.

This is not the place to discuss limit processes for defining the vector derivative (19.4) and proving the fundamental theorem (19.3). However, the method of simplices in [16] deserves mention, because it provides a practical approach to finite element approximations.

Now we are prepared to explain how GC generalizes Cartan's theory of differential forms. For $k \leq m$ a *differential k-form* $L = L(d^k x, x)$ on a manifold $\mathcal{M}^m$ is a multivector-valued k-form, that is, it is a linear function of the k-vector $d^k x$ at each point x. The simplest example is the volume element

$d^k x$, which is a k-vector-valued k-form. Another example is the $(m-1)$-form $d^{m-1}xF(x)$ on the right side of (19.3).

In Cartan's terminology, the *exterior differential* of the k-form L is a $(k+1)$-form dL defined here by

$$dL \equiv \dot{L}(d^{k+1}x \cdot \dot{\partial}) = L(d^{k+1}x \cdot \dot{\partial}, \dot{x}),$$

where the overdot indicates the variable differentiated. Cartan's abbreviated notation dL suppresses the dependence on the volume element $d^{k+1}x$ that is explicit in this definition. Note that the term "differential" as used here refers to the fact that the form is a linear function of a "volume element" intended to reside under an integral sign. In the next section we use the term "differential" in a different sense related to transformations. However, the two senses are intertwined when the transformation is applied to a form, which is just a particular kind of tensor.

Now we can express the *Fundamental Theorem of Geometric Calculus* in its most general form by the equation

$$\int_{\mathcal{M}} dL = \oint_{\partial\mathcal{M}} L. \tag{19.5}$$

This looks identical to the "*Generalized Stokes' Theorem*" in Cartan's calculus of differential forms. However, Cartan's forms are limited by being scalar-valued and lacking the complete algebraic structure of GC. More specifically, Cartan's theory is limited to functions of the form $L = \langle d^k xF(x)\rangle = (d^k x) \cdot F(x)$, where the angular brackets indicate scalar part and the center-dot applies if F is k-vector-valued. Accordingly, the exterior differential becomes

$$dL = \langle d^{k+1}x\ \partial F(x)\rangle = \langle d^{k+1}x\ \partial \wedge F\rangle = (d^{k+1}x) \cdot (\partial \wedge F).$$

Thus, Cartan's exterior differential is equivalent to the curl in GC, though its use in applications is more limited. When it is used to formulate Maxwell's equations, for example, the implicit volume element is just excess baggage, except when integration is intended.

The GC generalization to multivector-valued differential forms has profound applications. For example, it follows immediately from (19.3) that, with simple provisos,

$$\partial F = 0 \quad \Longleftrightarrow \quad \int_{\partial\mathcal{M}} d^{m-1}x\, F = 0.$$

For $m = 2$ this can be recognized as *Cauchy's Theorem* for complex variables, so it gives a straight-forward generalization of that theorem to higher dimensions. Similarly, a generalization of the justly famous *Cauchy Integral formula* can easily be derived from (19.5), as explained elsewhere [8, 5].

19.5 Mappings and Transformations

With the concept of vector derivative in hand, we are prepared to see how Geometric Calculus enables coordinate-free *transformations* of multivector fields on a given manifold or in *mappings* from one manifold to another. The power of this formalism is amply demonstrated in an elegant new approach to General Relativity called *Gauge Theory Gravity* [11, 1].

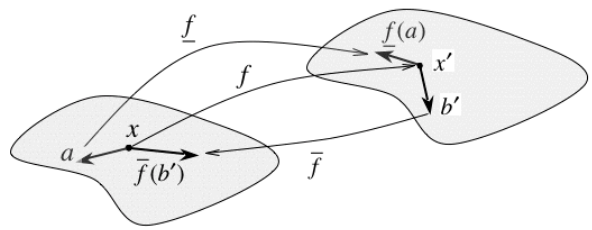

Fig. 19.3 Induced transformations of vector fields.

Let f be a an invertible diffeomorphism from one region of a given manifold to another, as expressed by

$$f : x \to x' = f(x), \qquad \text{so that} \quad x = f^{-1}(x').$$

This transformation induces a linear transformation $\underline{f}$ of the tangent space at each point called the *differential* of f. Accordingly, each vector field $a = a(x)$ undergoes a transformation (Fig. 19.3) defined by

$$\underline{f} : \; a = a(x) \;\; \to \;\; a' = \underline{f}(a) \equiv a \cdot \partial f, \qquad \text{so that} \quad a = \underline{f}^{-1}(a').$$

The *adjoint* $\overline{f}$ of the transformation is an induced linear transformation (Fig. 19.3) in the reverse direction:

$$\overline{f} : \; b' = b'(x') \;\; \to \;\; b = \overline{f}(b') \equiv \partial_x f(x) \cdot b'.$$

For applications one needs the theorem that the adjoint of the inverse transformation is the inverse of the adjoint:

$$\overline{f^{-1}} = \overline{f}^{\,-1} : b(x) \;\; \to \;\; b'(x') = \overline{f^{-1}} \; [b(f(x'))].$$

Note the complementary roles of directional derivative and gradient in the definitions of differential and adjoint. Also note that no notion of "differential as infinitesimal displacement" is involved.

To relate the GC approach to standard tensor calculus, consider a *rank-2 tensor* (field) $T(a, b')$ that is a linear function of vector fields a and b'. If these fields transform according to the differential and adjoint laws respectively, the tensor is said to be *contravariant* in the first argument and *covariant* in the second.

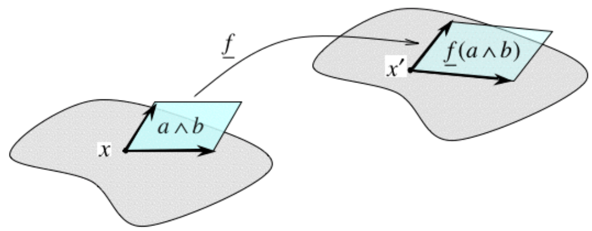

Fig. 19.4 Outermorphism of the differential.

The unique power of GC is manifest in the concept of *outermorphism*: the unique extension of a linear transformation defined on a vector space to *a linear transformation that preserves the outer product*, and hence defined on the entire GA generated by the vector space [8, 4]. For a pair of vector fields, the outermorphism of the differential gives us

$$\underline{f}:\ a\wedge b \ \rightarrow\ \underline{f}(a\wedge b)=\underline{f}(a)\wedge\underline{f}(b),$$

as illustrated in Fig. 19.4. By linearity, this property generalizes easily to the outermorphism of any multivector field [8]. In particular, it follows that the outermorphism of the pseudoscalar $I=I(x)$ is

$$\underline{f}:\ I\ \rightarrow\ \underline{f}(I)=J_f I' \quad \text{so that}\quad J_f=\det\underline{f}=I'^{-1}\underline{f}(I),$$

which shows that the Jacobian of the transformation, J_f, is just a scale factor induced by the outermorphism of the pseudoscalar.

A generalization of the familiar *chain rule* for differentiation is given by the transformation law for the vector derivative:

$$\overline{f}:\ \partial'\ \rightarrow\ \partial=\overline{f}(\partial') \qquad \text{or} \qquad \partial_x=\overline{f}(\partial_{x'}).$$

This implies invariance of the directional derivative:

$$a\cdot\partial=a\cdot\overline{f}(\partial')=\underline{f}(a)\cdot\partial'=a'\cdot\partial'. \tag{19.6}$$

Note that this applies whether a is a vector field or just a single vector in a given tangent space. For example, if $x=x(\tau)$ is a curve with tangent $\dot{x}=dx/d\tau$, then (19.6) implies invariance of the chain rule for differentiating fields:

$$\frac{d}{d\tau}=\dot{x}\cdot\partial_x=\dot{x}\cdot\overline{f}(\partial_{x'})=\underline{f}(\dot{x})\cdot\partial_{x'}=\dot{x}'\cdot\partial_{x'}.$$

Now we have all the necessary tools in hand for addressing the main subject of this article: *coordinate-free differential geometry.*

19.6 Shape and Curvature

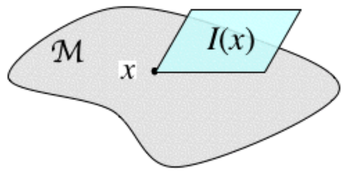

Fig. 19.5 Manifold pseudoscalar.

My purpose here is to explain how *the differential geometry of a given vector manifold* $\mathcal{M} = \{x\}$ *is completely determined by properties of its* pseudoscalar $I = I(x)$. For an *oriented manifold* the pseudoscalar is a single-valued field defined on the manifold. It can be visualized at each point (Fig. 19.5) as the tangent space (which it determines). Actually, as we have seen, it is a defining property of the manifold. For an unoriented manifold like a Möbius strip the pseudoscalar is doublevalued, as the orientation (algebraic sign) can be reversed by sliding it smoothly along a closed curve. But that is a minor point that will not concern us.

Let $a = a(x)$ be a *vector-valued function* defined on the mainfold. We say that it is a *vector field* if its values lie in the tangent space at each point of the manifold. This property is definitively determined by the pseudoscalar. Thus *projection* into the tangent space is a linear function defined by $P(a) \equiv (a \cdot I)I^{-1} \equiv a_{\|}$, while *rejection* from the tangent space is defined by $P_{\perp}(a) \equiv (a \wedge I)I^{-1} \equiv a_{\perp}$. Whence we derive the obvious result

$$a = (a \cdot I + a \wedge I)I^{-1} = P(a) + P_{\perp}(a) = a_{\|} + a_{\perp}.$$

Of course, a vector field has the tangency properties $a \wedge I = 0$ and $a = P(a)$.

The differential $P_b(a)$ of the manifold projection operator is given by straightforward differentiation:

$$P_b(a) \equiv b \cdot \dot{\partial}\dot{P}(a) = b \cdot \partial P(a) - P(b \cdot \partial a).$$

Note that $b \cdot \partial = [P(b)] \cdot \partial$, that is, the inner product of any vector with the vector derivative projects that vector into the tangent space, so differentials are always taken with respect to tangent vectors or vector fields.

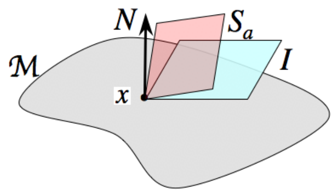

Fig. 19.6 Shape and Spur.

As we are not interested in the specific vector direction b, we can differentiate it out to get the *shape tensor*:

$$S(a) \equiv \dot{\partial}\dot{P}(a) = \partial_b P_b(a) = \dot{\partial} \wedge \dot{P}(a) + \dot{\partial} \cdot \dot{P}(a).$$

It is easy to prove the following theorems:

$$\dot{\partial} \wedge \dot{P}(a) = S(a_{\|}) \qquad \Rightarrow \qquad \dot{\partial} \wedge \dot{P}(a_{\perp}) = 0,$$

$$\dot{\partial} \cdot \dot{P}(a) = S(a_{\perp}) \qquad \Rightarrow \qquad \dot{\partial} \wedge \dot{P}(a_{\|}) = 0.$$

Consequently, we can decompose the shape tensor into bivector and scalar parts:

$$S(a) = S_a + N \cdot a,$$

where

$$S_a \equiv \dot{\partial} \wedge \dot{P}(a) = S[P(a)] = S(a_{\|}) \tag{19.7}$$

and

$$N \equiv \dot{P}(\dot{\partial}) = \partial_a S_a. \tag{19.8}$$

By virtue of (19.7) the *shape bivector* S_a could well be called the *curl* of the manifold $\mathcal{M}$. It follows that $P(S_a) = 0$, so the bivector-valued tensor S_a is not a field, as its values are not in the tangent algebra of $\mathcal{M}$.

The vector N is called the *spur* (of $\mathcal{M}$), see Fig. 19.6. It follows from (19.8) that $N \cdot P(a) = N \cdot a_{\|} = 0$, so N is not a vector field, as it is everywhere orthogonal to the tangent algebra of $\mathcal{M}$. As far as I know, the spur was not identified as a significant geometrical concept until it was first formulated in GC. We will not pursue it here. Rather, we aim to see how the shape tensor relates to standard concepts of differential geometry.

The shape bivector has a simple geometric interpretation with great intuitive appeal. It is easy to prove from its definition above that *the shape bivector is the rotational velocity of the pseudoscalar as it slides along the manifold*; formally,

$$S_a = I^{-1} a \cdot \partial I.$$

Alternatively,

$$\partial I = I S_a = I \times S_a.$$

where the symbol $\times$ denotes the commutator product and the last inequality is a consequence of $I \cdot S_a = 0$ and $I \wedge S_a = 0$.

The *curvature* of the manifold is given by the shape commutator, defined for vectors a and b by

$$C(a \wedge b) \equiv S_a \times S_b = P(S_a \times S_b) + P_\perp(S_a \times S_b). \tag{19.9}$$

The right side shows that the full curvature decomposes into distinct intrinsic and extrinsic parts. It can be proved that the intrinsic part is the usual *Riemann curvature*, which can accordingly be defined by

$$R(a \wedge b) \equiv P(S_a \times S_b).$$

Readers may be surprised that this simple expression does not involve the usual "coefficents of connexion." The moral is that the treatment of intrinsic geometry can be simplified by coordinating it with extrinsic geometry! – a striking claim that surely deserves close scrutiny. Supported by the power of GC, the shape tensor provides the means for investigating this claim.

Extension of the derivative concept to "covariant derivative" is at the heart of standard differential geometry. GC generalizes this to extension of the vector derivative $\partial = \partial_x$ to a *coderivative* $D = D_x$ defined, for action on any multivector-valued function $A = A(x)$, by

$$DA \equiv P(\partial A) = D \wedge A + D \cdot A.$$

It follows that

$$\partial A = DA + S(A),$$

where the *shape tensor* for A is given by

$$S(A) \equiv \dot{\partial}\dot{P}(A) = \dot{\partial} \wedge \dot{P}(A) + \dot{\partial} \cdot \dot{P}(A) = S(A_\parallel) + S(A_\perp).$$

For any tangent field $A = P(A) = A(x)$, the *cocurl* is given by

$$D \wedge A = P(\partial \wedge A) = \partial \wedge A - S(A),$$

while the *codivergence* is given by

$$\partial \cdot A = D \cdot A = [D \wedge (AI)]I^{-1}.$$

Many valuable differential identities can be derived from these definitions, such as

$$D \wedge D \wedge A = P(\partial \wedge \partial \wedge A) = 0, \qquad D \cdot (D \cdot A) = \partial \cdot (\partial \cdot A) = 0.$$

The equivalent of the covariant derivative is the *directional coderivative* (or *codifferential*) defined by

$$\delta_a A \equiv a \cdot DA \equiv P(a \cdot \partial A).$$

The commutator of codifferentials is determined by the intrinsic curvature:

$$(\delta_a \delta_b - \delta_b \delta_a) A = A \times R(a \wedge b)$$

The Riemann curvature is a bivector function of a bivector variable with the following (mostly well-known) properties of great importance in General Relativity Theory:

$$\begin{aligned}
\text{Symmetry:} \quad & (a \wedge b) \cdot R(c \wedge d) = (c \wedge d) \cdot R(a \wedge b) \\
\text{Ricci Identity:} \quad & a \cdot R(b \wedge c) + b \cdot R(c \wedge a) + c \cdot R(a \wedge b) = 0 \\
\text{Ricci tensor:} \quad & R(b) \equiv \partial_a R(a \wedge b) = -D \cdot S_b \\
\text{Scalar curvature:} \quad & R \equiv \partial_b R(b) = \partial \cdot N \\
\text{Bianchi identity:} \quad & \dot{D} \wedge \dot{R}(a \wedge b) = 0 \\
\text{Einstein tensor:} \quad & G(a) \equiv R(a) - \tfrac{1}{2} a R
\end{aligned}$$

This should suffice to clarify how the shape tensor relates to conventional formulations of differential geometry. For important examples of manifold geometry, we turn to the special case of manifolds that are hypersurfaces in a given manifold.

19.7 Hypersurfaces and classical geometry

The shape tensor generalizes the original approach to classical differential geometry developed by Gauss, who characterized *surfaces* (2d manifolds) in terms of their normals. The straightforward generalization of his approach to hypersurfaces of any dimension has been formulated in modern terms by [10]. Let us see how to do it with GC.

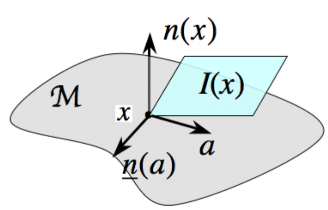

Fig. 19.7 The normal and its differential.

Let $\mathcal{M} = \mathcal{M}_m$ be an m-dimensional hypersurface in $\mathbb{E}_{m+1}$. Let $i = \langle i \rangle_{m+1}$ = constant be the unit pseudoscalar for $\mathbb{E}_{m+1}$. Then the pseudoscalar for $\mathcal{M}$ is given by $I = ni$, where $n = n(x)$ is the *unit normal.* The function $n = n(x)$ is often called the "Gauss map" to support the intuition that it is a mapping of the manifold onto a unit sphere. Instead, we describe the normal sliding on the hypersurface in terms of its differential $\underline{n}(a) = a \cdot \partial n$. Then the shape of the hypersurface reduces to a function of the normal and its differential, see Fig. 19.7:

$$S_a = I^{-1} a \cdot \partial I = n\underline{n}(a) = n \wedge \underline{n}(a)$$

It is now straightforward to reduce geometric quantities in the previous section to functions of the normal and its differential:

$$\begin{aligned}
\text{Curvature:} \quad & R(a \wedge b) = P(S_a \times S_b) = \underline{n}(a) \wedge \underline{n}(b) = \underline{n}(a \wedge b) \\
\text{Mean Curvature:} \quad & H \equiv \frac{1}{m} \partial_a \underline{n}(a), \text{with } \partial_a \underline{n}(a) = \text{tr } \underline{n} = \partial \cdot n = -n \cdot N \\
\text{Scalar Curvature:} \quad & R = (\partial_b \wedge \partial_a) \cdot \underline{n}(a) \wedge \underline{n}(b) = (\text{tr } \underline{n})^2 - \text{tr } \underline{n}^2 \\
\text{Gaussian Curvature:} \quad & \kappa = I^{-1} \cdot \underline{n}(I)
\end{aligned}$$

For $m = 2$ the Gaussian curvature can be written $\kappa = I^{-1} \cdot R(I)$, so it is equivalent to the Riemann curvature, which has only one component. It is worth noting that the extrinsic component of the curvature (19.9) gives the classical Codazzi-Mainardi equations for extrinsic geometry of a hypersurface, but we will no go into that.

The rest of this section is devoted to surfaces in $\mathbb{E}_3$, since that is the case of greatest practical interest to engineering and computer graphics. Interested readers are invited to compare the present approach to the classical treatment in [17] using vector calculus. Rather than review examples in [8], let me discuss an important classical example with a new twist.

According to Coulomb's law, the electric potential of a finite line charge of length L and charge density λ is given by the integral

$$V(\mathbf{x}) = \int \frac{k dq(s)}{|\mathbf{x} - \mathbf{x}'(s)|} = \int_{-L/2}^{L/2} \frac{k\lambda ds}{|\mathbf{x} - s\mathbf{e}|} \equiv k\lambda\varphi(\mathbf{x}).$$

Remarkably, a simple expression for the value of this integral was overlooked until recently when Rowley [14] discovered

$$\varphi(\mathbf{x}) = \ln\left(\frac{r_+ + r_- + L}{r_+ + r_- - L}\right), \quad \text{where} \quad r_\pm = |\mathbf{r}_\pm| = |\mathbf{x} \pm \tfrac{1}{2}L\mathbf{e}|.$$

The equipotentials compose a family of *confocal ellipsoids* (Fig. 19.8). The *eccentricity* ε and *directrix* d of each ellipsoid is given by $\varepsilon \equiv \tanh(\varphi/2) < 1$, so that

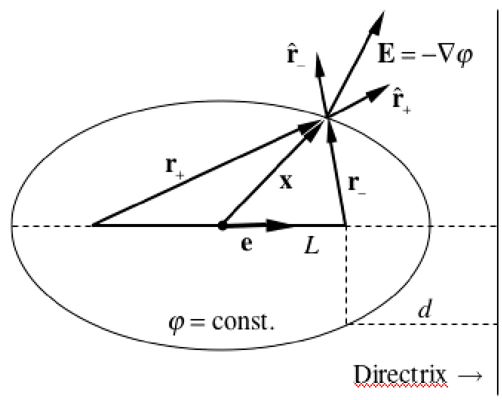

Fig. 19.8 Ellipsoidal equipotentials.

$$\frac{1+\varepsilon}{1-\varepsilon} = e^{\varphi} = \frac{r_+ + r_- + L}{r_+ + r_- - L}$$

$$d = \frac{(r_+ + r_-)^2 - L^2}{2L} = (\varepsilon^{-2} - 1)\frac{L}{2},$$

$$r_+ + r_- = \frac{L}{\varepsilon}.$$

The electric field (for $k\lambda = 1$) is given by

$$\mathbf{E} = -\nabla\varphi = \frac{\hat{\mathbf{r}}_+ + \hat{\mathbf{r}}_-}{d}.$$

Here is the surprising new geometric fact that Rowley discovered: The unit normal $\mathbf{n}$ at each point of an ellipse or ellipsoid is given by

$$\hat{\mathbf{r}}_+ + \hat{\mathbf{r}}_- = \nabla(r_+ + r_-) = \Lambda\,\mathbf{n} \qquad \Lambda^2 = (\hat{\mathbf{r}}_+ + \hat{\mathbf{r}}_-)^2 = 2(1 + \hat{\mathbf{r}}_+ \cdot \hat{\mathbf{r}}_-).$$

Of course, the difference of the unit coradius vectors is a tangent vector $\hat{\mathbf{r}}_+ - \hat{\mathbf{r}}_- \equiv 2\mathbf{t}$. All this gives us a simple and perspicuous expression for the differential of the normal: For any tangent vector $\mathbf{a} = P(\mathbf{a})$,

$$\underline{\mathbf{n}}(\mathbf{a}) \equiv \mathbf{a}\cdot\nabla\mathbf{n} = \lambda[\mathbf{a} - (\mathbf{a}\cdot\mathbf{t})\mathbf{t}], \quad \text{with} \quad \lambda = \left(\frac{1}{r_+} + \frac{1}{r_-}\right)\frac{1}{\Lambda} = \frac{L}{\varepsilon r_+ r_- \Lambda}.$$

Note that the tangent vector $\mathbf{t}$ is an eigenvector of the differential $\underline{\mathbf{n}}$.

It is now a simple matter to compute all geometric properties of an ellipsoid of revolution using the apparatus developed above. Since ellipsoids have many practical applications and the present approach is new, it is worth recording the main results for future reference. For an m-dimensional ellipsoid of revolution we find

$$\begin{aligned}\operatorname{tr}\underline{\mathbf{n}} &= \lambda(m - \mathbf{t}^2)\\ \underline{\mathbf{n}}^2(\mathbf{a}) &= \lambda[\mathbf{n}(\mathbf{a}) - \mathbf{t}\mathbf{t}\cdot\underline{\mathbf{n}}(\mathbf{a})] = \lambda^2[\mathbf{a} + (\mathbf{a}\cdot\mathbf{t})(\mathbf{t}^2 - 2)\mathbf{t}]\\ \operatorname{tr}\underline{\mathbf{n}}^2 &= \lambda^2[m + (\mathbf{t}^2 - 2)\mathbf{t}^2] = \frac{\lambda^2}{4}\left(4m - 3 - 5\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}}_- + 2(\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}}_-)^2\right).\end{aligned}$$

Therefore:

$$\begin{aligned}\text{Shape:}\quad & S_{\mathbf{a}} = \mathbf{n}\underline{\mathbf{n}}(\mathbf{a}) = \lambda\mathbf{n}[\mathbf{a} - (\mathbf{a}\cdot\mathbf{t})\mathbf{t}]\\ \text{Curvature:}\quad & R(\mathbf{a}\wedge\mathbf{b}) = \underline{\mathbf{n}}(\mathbf{a})\wedge\underline{\mathbf{n}}(\mathbf{b}) = \underline{\mathbf{n}}(\mathbf{a}\wedge\mathbf{b})\\ & = \lambda^2\left(\mathbf{a}\wedge\mathbf{b} - \mathbf{t}\wedge[\mathbf{t}\cdot(\mathbf{a}\wedge\mathbf{b})]\right)\\ \text{Mean Curvature:}\quad & H \equiv \frac{1}{m}\operatorname{tr}\underline{\mathbf{n}} = \frac{\lambda}{m}(m - \mathbf{t}^2).\end{aligned}$$

For the case $m = 2$ we have the particular results

$$\begin{aligned}\text{Curvature:}\quad & R(\mathbf{a}\wedge\mathbf{b}) = \kappa\mathbf{a}\wedge\mathbf{b}\\ \text{Gaussian Curvature:}\quad & \kappa = \lambda^2(1 - \mathbf{t}^2) = \tfrac{1}{2}\lambda^2(1 + \hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}}_-)\\ \text{Mean Curvature:}\quad & H = \tfrac{1}{2}\operatorname{tr}\underline{\mathbf{n}} = \lambda(1 - \tfrac{1}{2}\mathbf{t}^2) = \tfrac{1}{2}\lambda\left(\frac{3}{2} + \hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}}_-\right).\end{aligned}$$

All this has some obvious generalizations, for example, to a general ellipsoid with an orthonormal set of tangent vectors, which, like $\mathbf{t}$, are eigenvectors of the differential $\underline{\mathbf{n}}$.

Now let us turn to a general question of great interest and utility: *What is the "shape" of a curve embedded in a manifold?* Shape and curvature are not defined for a curve, because it is a one-dimensional manifold. Instead, shape and curvature bivectors are replaced by the Darboux bivector [8], which completely characterizes the geometry of the curve. Let us address our question for curves in $\mathbb{E}_3$ embedded in some surface, since that is the case of greatest practical interest. The GC apparatus we are using makes generalization to higher dimensions (and even mixed signature) fairly straightforward. In deference to that possibility, we drop the convention of boldface type for vectors in Euclidean space.

Let $x = x(s)$ be a curve with arc length s. Then its "velocity" is a unit tangent vector $v = dx/ds \equiv \dot{x}$. All derivatives of v are determined by the *Darboux bivector* Ω_v. In particular, the acceleration is given by the Frenet equation

$$\dot{v} = \Omega_v \cdot v.$$

Its magnitude is called the *first curvature* $\kappa_\tau = |\dot{v}|$.

The condition that the curve is embedded in a surface with normal $n = n(x)$ is that $v = P(v)$ is a tangent vector and

$$S_v = P_\perp(\Omega_v) = n\underline{n}(v) = n \wedge \underline{n}(v).$$

This decomposes the "Darboux" into two parts:

$$\Omega_v = S_v + \omega_v = P_\perp(\Omega_v) + P(\Omega_v),$$

where $\omega_v = P(\Omega_v)$ is the rotation rate of the curve within the surface. This decomposition can be characterized by two *bending invariants* [17], the *normal curvature* $\kappa_v = v \cdot \underline{n}(v)$ and the *geodesic (tangential) curvature* $\kappa_g = -I \cdot \omega_v = u \cdot \omega_v \cdot v = u \cdot \dot{v}$, where $u = Iv$ is a unit vector orthogonal to v. Obviously, $\kappa_g^2 = -S_v^2$ and $\kappa_g^2 = -\omega_v^2$. This completes our answer to the question about the "shape" of an embedded curve.

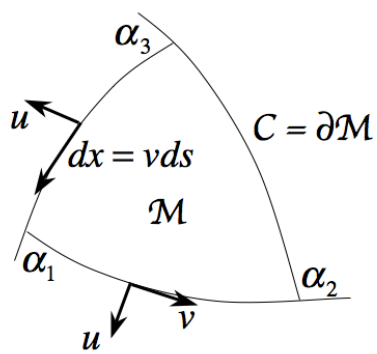

Fig. 19.9 Triangular domain for the Gauss-Bonnet formula.

We can use what we have just learned to understand the beautiful and profound *Gauss-Bonnet Formula*:

$$\int_{\mathcal{M}} \kappa dA + \oint_{\mathcal{C}} \kappa_g ds + \sum_i \alpha_i = 2\pi. \tag{19.10}$$

This applies to any simply connected surface $\mathcal{M}$ bounded by a piecewise differentiable closed curve $\mathcal{C} = \partial\mathcal{M}$ with outer normal $u = Iv$ and exterior angles α_i, as illustrated in Fig. 19.9. As before, $\kappa = I^{-1}R(I)$ is the Gaussian curvature and $\kappa_g = u \cdot \dot{v} = -\dot{u} \cdot \dot{x}$ is the geodesic curvature. In GC terms, using the Riemann curvature $R(d^2x) = \kappa d^2x$ with directed area element $d^2x = IdA$, the formula can be written

$$\int_{\mathcal{M}} I^{-1}R(d^2x) - \oint_{\mathcal{C}} \dot{u} \cdot dx + \sum_i \alpha_i = 2\pi. \tag{19.11}$$

Proof of the Gauss-Bonnet formula is a nice application of the Fundamental Theorem [17]. Generalization of the formula to higher dimensions is highly non-trivial [8], and it involves the Riemann curvature in the way it appears in (19.11). No doubt there is more to be learned about this generalization and variations on the theme.

Now consider an important special case. The bounding curves are geodesics if $\kappa_g = 0$, and the figure in Fig. 19.9 is a *geodesic triangle*. For a sphere of radius r the Gaussian curvature is r^2; whence the first term in (19.10)) and (19.11)) is the solid angle subtended by the region $\mathcal{M}$. An elegant expression for this solid angle in terms of the vectorial endpoints is derived in [8], which uses GA to describe the geometry of human body movement. Therein is discussed the amazing fact that the human eye has learned to implement this theorem to keep the retinal image upright in saccadic motion. That is the import of the psychophysical discovery known as *Listing's Law.*

19.8 Challenges

Let me conclude this review with a few challenges for further development of the theory and applications.

- **Extension to Conformal Geometric Algebra.** The concept of vector manifold is so general that there should be no problem in applying it to the case when all points are null vectors as required for CGA. I would recommend concentrating first on the geometry of hypersurfaces using the conformal split [7] with the normal at each point x given by the unit bivector $E = x \wedge e_\infty$.
- **Finite Element Differential Geometry.** There is an abundant literature on this subject with many examples worth translating into GA and CGA. Reference [16] should be especially helpful for discrete versions of the vector derivative and fundamental theorem. *Regge Calculus* is an elegant approach to discretizing Riemannian geometry developed for applications to General Relativity [13, 12]. Translation and adaptation to GC should be fairly easy and enlightening. Applications to engineering and computer science as well as physics look promising.
- **Geometry of Movement.** Using CGA to rework and extend the approach in [6] has great potential for robotics as well as biomechanics.
- **Elasticity.** The geometry of material media, including constitutive relations as well as stresses, strains and deformations should be a fertile domain for GC applications.
- **Tangent cones for discontinuities.** So far our approach to differential geometry has ignored discontinuities and singularities of all kinds. GC is well suited to handle such issues, especially in concert with the finite element approach to geometry proposed above. But here is another approach

worth investigating. My father developed the concept of *tangent cone* as a portion of the tangent space at a point wherein convergence to a limit obtains, and he applied it with great success to rigorous treatment of singularities in calculus of variations [9]. We have characterized the geometry of a manifold by properties of the pseudoscalar for the tangent space. My suggestion is to meld this notion with the tangent cone idea by using a more general multivector to describe limit structure in the tangent space at points that lie on creases, edges, corners and other discontinuities. I regard this as a hard problem, because it is not well-defined and I don't really know how to approach it.

References

1. C. Doran and A. Lasenby. *Geometric Algebra for Physicists.* Cambridge: The University Press, 2003.
2. L. Dorst, D. Fontijne, and S. Mann. *Geometric Algebra for Computer Science.* San Francisco: Morgan Kaufmann, 2007.
3. D. Hestenes. *Space-Time Algebra.* New York: Gordon and Breach, 1966.
4. D. Hestenes. The design of linear algebra and geometry. *Acta Applicandae Mathematicae*, 23:65–93, 1991.
5. D. Hestenes. Differential forms in geometric calculus. In F. Brackx and et al., editors, *Clifford Algebras and their Applications in Mathematical Physics*, pages 269–285. Kluwer: Dordrecht/Boston, 1993.
6. D. Hestenes. Invariant body kinematics: I. saccadic and compensatory eye movements and II. reaching and neurogeometry. *Neural Networks*, 7:65–88, 1994.
7. D. Hestenes. New tools for computational geometry and rejuvenation of screw theory. In E. Bayro-Corrochano and G. Scheuermann, editors, *Geometric Algebra Computing for Engineering and Computer Science.* London: Springer Verlag, 2009.
8. D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus, a unified language for mathematics and physics, Fourth printing 1999.* Kluwer: Dordrecht/Boston, 1984.
9. M. R. Hestenes. *Calculus of Variations and Optimal Control Theory.* Wiley, New York, 1966.
10. N. Hicks. *Notes on Differential Geometry.* New York: Van Nostrand, 1965.
11. A. Lasenby, C. Doran, and S. Gull. Gravity, gauge theories and geometric algebra. *Phil. Trans. R. So. London A*, 356:161, 2000.
12. W. Miller. The geometrodynamic content of the Regge equations as illuminated by the boundary of a boundary principle. *Foundations of Physics*, 16(2):143–169, 1986.
13. T. Regge. General relativity without coordinates. *Nuovo Cimento*, 19:558–571, 1961.
14. R. Rowley. Finite line of charge. *American Journal of Physics*, 74:1120–1125, 2006.
15. G. Sobczyk. Killing vectors and embedding of exact solutions in general relativity. In J. Chisholm and A. Common, editors, *Clifford Algebras and their Applications in Mathematical Physics*, pages 227–244. Dordrecht/Boston: Reidel, 1986.
16. G. Sobczyk. Simplicial calculus with geometric algebra. In A. Micali, R. Boudet, and J. Helmstetter, editors, *Clifford Algebras and their Applications in Mathematical Physics*, pages 227–244. Dordrecht/Boston: Kluwer, 1992.
17. D. Struik. *Lectures on Classical Differential Geometry.* Reading: Addison-Wesley, 1961.

Exercises

1. Suppose that the geodesic triangle in Fig. 19.9 lies on a unit sphere with vertices at $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ so $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = 1$. Parallel transfer of a vector $\mathbf{p}$ around the triangle can be calculated as follows: The tangent vector $\mathbf{p}$ at $\mathbf{a}$ is transferred to a tangent vector $A\,\mathbf{p}\,A^{-1}$ at $\mathbf{b}$ by the spinor $A = 1 + \mathbf{ba}$. It can subsequently be transferred to the point $\mathbf{c}$ by $B = 1 + \mathbf{cb}$ and back to $\mathbf{a}$ by $C = 1 + \mathbf{ac}$. The net result is rotation by a spinor $T = CBA$. Show that

$$\tfrac{1}{2}T = 1 + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{a}\,(\mathbf{c}\wedge\mathbf{b}\wedge\mathbf{a}),$$

so $\mathbf{p}$ is rotated about the axis $\mathbf{a}$ through an angle ϕ given by

$$\tan(\tfrac{1}{2}\phi) = \frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{1 + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}}$$

How does this angle relate to the area of the triangle?

2. Use the result of the previous exercise to explain how the eye must rotate during saccades in order to keep the image on the retina erect. See [6] for details.

3. Generalize Rowleys potential $\phi(\mathbf{x})$ for an ellipsoid of revolution to an ellipsoid with axes $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Calculate the shape and curvature tensors.

4. Find an explicit expression for the Darboux bivector of a geodesic on an ellipsoid. Calculate its normal and geodesic curvatures. How do these relate to the curvatures of the ellipsoid?

5. Apply the Fundamental Theorem of Geometric Calulus to prove the Gauss-Bonnet Formula (19.10).