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# SPINOR PARTICLE MECHANICS

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**Abstract.** Geometric Algebra makes it possible to formulate simple spinor equations of motion for classical particles and rigid bodies. In the Newtonian case, these equations have proven their value by simplifying orbital computations. The relativistic case is not so well known, but it has new and surprising features worth exploiting, including close connections to quantum mechanical equations. The current status of spinor particle mechanics is reviewed, and directions for extention are pointed out.

**Key words:** Spinor, Geometric algebra, Spacetime algebra, orbital mechanics, relativity.

## 1. INTRODUCTION

Geometric Algebra brings a host of new insights and computational techniques to mathematics and physics. Many of them follow from a clarification and simplification of the spinor concept. Few mathematicians and physicists realize even yet that spinor methods can breathe new life into that most venerable branch of mathematical physics—particle mechanics. This paper shows how.

After a brief introduction to Geometric Algebra in Section 2, Section 3 highlights some advantages of the Algebra in Newtonian orbital mechanics, in particular, conceptual and computational simplifications that have recently been incorporated into software for government and commercial space programs.

Section 4 introduces the Geometric Algebra of spacetime. Then Section 5 reviews the remarkable spinor solution to the relativistic Coulomb problem and suggests extensions of the method. Most notable is the extension of the nonrelativistic Kustannheimo-Stiefel equation to the relativistic case.

Finally, Section 6 outlines a spinor approach to gravitational motion and precession in a new gauge theory of gravity.

## 2. GEOMETRIC ALGEBRA OF EUCLIDEAN 3-SPACE

The material in this section and the next has been treated at length in [1], so we can be brief. We represent positions in the "Physical Space" of Newtonian mechanics by vectors in a 3-D Euclidean space $\mathcal{E}_3$. The *Geometric Algebra* $\mathcal{G}_3$ for this space is an associative algebra over the real numbers generated by defining a *geometric product* on $\mathcal{E}_3$ with the property that the square of any nonzero vector is a non-negative scalar. Thus, for any vector $\mathbf{a}$,

$$\mathbf{a}^2 = |\,\mathbf{a}\,|^2 \geq 0\,, \tag{2.1}$$

where the scalar $|\,\mathbf{a}\,| \geq 0$ is the *magnitude* or *modulus* of $\mathbf{a}$. It follows that the *geometric product* $\mathbf{ab}$ of two vectors admits the decomposition

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\,, \tag{2.2}$$

where

$$\mathbf{a}\cdot\mathbf{b} = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) \tag{2.3}$$

is the usual Euclidean *inner product*, and

$$\mathbf{a}\wedge\mathbf{b} = \tfrac{1}{2}(\mathbf{ab} - \mathbf{ba}) \tag{2.4}$$

is the *outer product.*

Let $\{\boldsymbol{\sigma}_k;\, k = 1, 2, 3\}$ be a "right-handed" orthonormal basis in $\mathcal{E}_3$. It is related to the *righthanded unit pseudoscalar* $i$ by

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i\,. \tag{2.5}$$

*Duality* in $\mathcal{G}_3$ is defined as multiplication by $i$. This can be used to define the cross product $\mathbf{a}\times\mathbf{b}$ of conventional vector algebra as the dual of the outer product; thus

$$\mathbf{a}\times\mathbf{b} = i(\mathbf{b}\wedge\mathbf{a}) = -i(\mathbf{a}\wedge\mathbf{b})\,. \tag{2.6}$$

A generic element $M$ of $\mathcal{G}_3$, called a *multivector*, can be written in the expanded form

$$M = \alpha + \mathbf{a} + i\mathbf{b} + i\beta\,, \tag{2.7}$$

where $\alpha$ and $\beta$ are scalars, and $\mathbf{a}$ and $\mathbf{b}$ are vectors. The *reverse* $M^\dagger$ of $M$ can be defined by

$$M^\dagger = \alpha + \mathbf{a} - i\mathbf{b} - i\beta\,. \tag{2.8}$$

The *modulus* of $M$ is a non-negative (real) scalar $|\,M\,|$ defined by

$$|\,M\,|^2 = \langle\, MM^\dagger\,\rangle = \alpha^2 + \mathbf{a}^2 + \mathbf{b}^2 + \beta^2\,, \tag{2.9}$$

where $\langle\ldots\rangle$ denotes *scalar part.*

A multivector $U$ with the expanded form

$$U = \alpha + i\mathbf{b} \tag{2.10}$$

is a *quaternion.* We refer to it as a *spinor* if it is used to represent a rotation, as in

$$\mathbf{r} = U\boldsymbol{\sigma}_1 U^\dagger = r\hat{\mathbf{r}}\,, \tag{2.11}$$

where $\hat{\mathbf{r}}$ is a unit vector. This equation determines a parameterization of vectors in $\mathcal{E}_3$ by spinors. It describes a rotation of a fixed "reference vector" $\boldsymbol{\sigma}_1$ into the direction $\hat{\mathbf{r}}$, along with a dilation by

$$r = |\,\mathbf{r}\,| = |\,U\,|^2\,. \tag{2.12}$$

When the spinor $U$ is normalized to unity, it is called a *rotor.*

## 3. NEWTONIAN ORBITAL MECHANICS

Let $\mathbf{r} = \mathbf{r}(\tau)$ be the orbit of a particle (of unit mass) with the Newtonian equation of motion

$$\ddot{\mathbf{r}} = -\frac{k\hat{\mathbf{r}}}{r^2} + \mathbf{f}\,, \tag{3.1}$$

where the overdot indicates differentiation with respect to time $\tau$, which, for the nonrelativistic case, we identify with *coordinate time* $t$. The first term on the right side of (3.1) represents either the Coulomb force of a fixed point charge or the Newtonian graviational force of a point mass, depending on the coupling constant $k$. We will refer to $\mathbf{f}$ as the *perturbing force*.

The *angular momentum* (bivector) $\mathbf{L}$ is defined by

$$\mathbf{L} \equiv \mathbf{r} \wedge \dot{\mathbf{r}} = r^2 \hat{\mathbf{r}}\dot{\hat{\mathbf{r}}}\,. \tag{3.2}$$

We can solve (3.2) for

$$\dot{\hat{\mathbf{r}}} = \frac{\hat{\mathbf{r}}\mathbf{L}}{r^2} = -\frac{\mathbf{L}\hat{\mathbf{r}}}{r^2}\,. \tag{3.3}$$

This determines a decomposition of the velocity $\dot{\mathbf{r}}$ into radial and rotational parts:

$$\dot{\mathbf{r}} = \left(\dot{r} + \frac{\mathbf{L}}{r}\right)\hat{\mathbf{r}}\,, \tag{3.4}$$

with a corresponding decomposition of the kinetic energy given by

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + \frac{\ell^2}{r^2}\,, \tag{3.5}$$

where $\ell \equiv |\,\mathbf{L}\,|$.

From (3.1) we obtain

$$\dot{\mathbf{L}} = \mathbf{r} \wedge \mathbf{f}\,. \tag{3.6}$$

For the time being we assume that $\mathbf{f} = 0$, so that

$$\mathbf{L} = \ell\,\mathbf{i} \tag{3.7}$$

is a constant of the motion specifying a unit bivector $\mathbf{i}$ for the orbital plane. Multiplying (3.1) by $\mathbf{L}$ and using (3.3) we find immediately that

$$\mathbf{L}\mathbf{v} - k\hat{\mathbf{r}} = \boldsymbol{\epsilon} \tag{3.8}$$

is another constant of the motion. This constant is peculiar to the inverse square central force. It determines the direction $\hat{\boldsymbol{\epsilon}}$ of the orbit's major axis and the magnitude $\epsilon = |\,\boldsymbol{\epsilon}\,|$ of its eccentricity, so it is appropriately dubbed the *eccentricity vector*.

From the two constants of motion $\mathbf{L}$ and $\boldsymbol{\epsilon}$, all properties of the orbit can be found algebraically without integrations. Geometric algebra facilitates the algebraic manipulations considerably. To illustrate this, we derive the deflection formula for Coulomb scattering. Let $\mathbf{v} = \dot{\mathbf{r}}$ be the asymptotic initial velocity at $\tau = -\infty$, and let $\mathbf{v}_0 = \dot{\mathbf{r}}_0$ be the velocity at the point $\mathbf{r}_0$ of closest approach (*pericenter*). Then we can write

$$\mathbf{L} = \mathbf{b}\mathbf{v} = \mathbf{r}_0\mathbf{v}_0 = \ell\,\mathbf{i}\,, \tag{3.9}$$

where $b = |\,\mathbf{b}\,|$ is the *impact parameter*, so

$$\ell \equiv |\,\mathbf{L}\,| = bv = r_0 v_0\,. \tag{3.10}$$

The scattering angle $\Theta$ is defined by

$$\hat{\mathbf{v}}_0\hat{\mathbf{v}} = \hat{\mathbf{r}}_0\hat{\mathbf{b}} = e^{\mathbf{i}\Theta/2}\,. \tag{3.11}$$

From (3.8) we obtain

$$\mathbf{L}\mathbf{v} - k\hat{\mathbf{r}} = \mathbf{L}\mathbf{v}_0 - k\hat{\mathbf{r}}_0\,. \tag{3.12}$$

Asymptotically $\hat{\mathbf{v}} = -\hat{\mathbf{r}}$ so, with the help of (3.9), this gives us

$$(\mathbf{L}v + k)\hat{\mathbf{v}} = (\mathbf{L}v_0 + k\mathbf{i})\hat{\mathbf{v}}_0\,,$$

which can be solved for

$$\hat{\mathbf{v}}_0\hat{\mathbf{v}} = \frac{\ell v - \mathbf{i}k}{\ell v_0 + k} = e^{\mathbf{i}\Theta/2}\,. \tag{3.13}$$

This gives us immediately the famous deflection formula for Rutherford scattering:

$$\operatorname{ctn}\frac{\Theta}{2} = \frac{bv^2}{-k} = \frac{2Eb}{-k}\,, \tag{3.14}$$

where $E$ is the energy of the system, and the sign of the angle distinguishes between attractive and repulsive forces. The derivation here is a slight simplification of the one in [1], where more details are given.

Now we turn to a recent spinor formulation of Newtonian mechanics [1] with surprising implications. We use (2.11) and (2.12) to express the position vector $\mathbf{r}$ as a function of a spinor $U$, and we derive an equation of motion for $U$ to replace Newton's equation of motion for $\mathbf{r}$. Using the anticommutivity of $\mathbf{L}$ with $\mathbf{r}$ in (3.3), from (3.4) and (2.11) we derive

$$2\dot{U}U^{-1} = \dot{\mathbf{r}}\mathbf{r}^{-1} = r^{-1}\dot{r} - r^{-2}\mathbf{L}\,. \tag{3.15}$$

This can be solved for

$$\dot{\mathbf{r}} = 2\dot{U}\boldsymbol{\sigma}_1 U^\dagger \tag{3.16}$$

or

$$r\dot{U} = \frac{dU}{ds} = \tfrac{1}{2}\,\dot{\mathbf{r}}\,U\boldsymbol{\sigma}_1\,, \tag{3.17}$$

where a new parameter $s$ has been defined by the condition

$$\frac{d\tau}{ds} = r = |\,U\,|^2\,. \tag{3.18}$$

Differentiating (3.17) we obtain

$$2\frac{d^2U}{ds^2} = (\ddot{\mathbf{r}}\mathbf{r} + \tfrac{1}{2}\,\dot{\mathbf{r}}^2)U\,. \tag{3.19}$$

From (3.1) we get

$$\ddot{\mathbf{r}}\mathbf{r} + \tfrac{1}{2}\,\dot{\mathbf{r}}^2 = E + \mathbf{f}\,\mathbf{r}\,,$$

where

$$E = \tfrac{1}{2}\dot{\mathbf{r}}^2 - \frac{k}{r} = |\,U\,|^{-2}\left(2\left|\frac{dU}{ds}\right|^2 - k\right). \tag{3.20}$$

Thus we obtain the desired equation of motion for $U$:

$$2\frac{d^2U}{ds^2} - EU = \mathbf{f}\,\mathbf{r}U = |\,U\,|^2\mathbf{f}\,U\boldsymbol{\sigma}_1\,. \tag{3.21}$$

This is known as the *Kustaanheimo-Stiefel (K-S) equation*. It is most notable for eliminating the singularity at $r = 0$ and for linearizing the equation of motion for the Kepler-Coulomb problem.

For bound orbits ($E < 0$) with $\mathbf{f} = 0$, this reduces to the 2D simple harmonic oscillatory equation, with the solution

$$U = \alpha_0 \cos\big(\frac{\omega s}{2}\big) + \mathbf{i}\beta_0 \sin\big(\frac{\omega s}{2}\big)\,, \tag{3.22}$$

where

$$\omega^2 = 2|\,E\,|\,. \tag{3.23}$$

Hence (3.18) gives us

$$r = \tfrac{1}{2}(\alpha_0^2 + \beta_0^2) + \tfrac{1}{2}(\alpha_0^2 - \beta_0^2)\cos(\omega s) \tag{3.24}$$

and

$$\tau = \tfrac{1}{2}(\alpha_0^2 + \beta_0^2) + \frac{1}{2\omega}\,(\alpha_0^2 - \beta_0^2)\sin(\omega s)\,. \tag{3.25}$$

Note that (3.24) solves Kepler's problem relating coordinate time to position on the orbit.

When perturbing forces are included, the K-S equation (3.21) has significant computational advantages in applications to orbital mechanics. Vrbik [2] has refined the numerical techniques for using it to compute the effects of perturbing forces. Strom [3] has used geometric algebra to improve navigation software for the government and commercial space programs.

## 4. SPACETIME ALGEBRA

Spacetime Algebra (STA) has been fully discussed in many places, of which [4] and [5] are closest to our interests here. For that reason, we mention only the barest essentials needed for spinor mechanics.

STA is the Geometric Algebra of Minkowski Spacetime $\mathcal{M}_4$. As in any Geometric Algebra, the geometric product of two vectors $a$ and $b$ in $\mathcal{M}_4$ admits the decomposition

$$ab = a \cdot b + a \wedge b\,, \tag{4.1}$$

where $a \cdot b$ is the usual Minkowski *inner product*. We use the signature for which $a^2 = a \cdot a$ is positive for a timelike vector.

Any (future-pointing) timelike unit vector $\gamma_0$ determines a unique inertial system with a "split" of spacetime into one time dimension and three spatial dimensions.

Such a *spacetime split* is most simply expressed by the geometric product as follows: For each spacetime point (or event) represented by a vector $x$ in $\mathcal{M}_4$, the split is specified by

$$x\gamma_0 = t + \mathbf{r}\,, \tag{4.2a}$$

where

$$t = x \cdot \gamma_0 \tag{4.2b}$$

is the time of the event (in natural units) and

$$\mathbf{r} = x \wedge \gamma_0 \tag{4.2c}$$

represents the position of the event with respect to the inertial system determined by $\gamma_0$. Though $x \wedge \gamma_0$ is a bivector in spacetime, we can identify $\mathbf{r}$ as a position vector in the Euclidean space $\mathcal{E}_3$ of the preceding sections.

We can extend $\gamma_0$ to a complete orthonormal basis $\{\gamma_\mu;\ \mu = 0, 1, 2, 3,\}$ for $\mathcal{M}_4$, with

$$\gamma_0^2 = 1 \quad \text{and} \quad \gamma_k^2 = -1 \text{ for } k = 1, 2, 3\,. \tag{4.3}$$

In comformity with (4.2c), we note that the

$$\boldsymbol{\sigma}_k = \gamma_k \wedge \gamma_0 = \gamma_k\gamma_0 \tag{4.4}$$

forms a basis for $\mathcal{E}_3$. Indeed, we find that, in accord with (2.5),

$$\gamma_0\gamma_1\gamma_2\gamma_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i \tag{4.5}$$

defines a unit *pseudoscalar* for both STA and $\mathcal{G}_3$. Thus, $\mathcal{G}_3$ is a subalgebra of STA. A generic multivector $M$ in STA can be written in the expanded form

$$M = \alpha + a + F + ib + i\beta\,, \tag{4.6}$$

where $\alpha$ and $\beta$ are scalars, and $a$ and $b$ are vectors, and $F$ is a bivector. The *reverse* $\widetilde{M}$ of $M$ is defined by

$$\widetilde{M} = \alpha + a - F - ib + i\beta\,. \tag{4.7}$$

Its relation to the reverse defined in $\mathcal{G}_3$ by (2.8) is

$$M^\dagger = \gamma_0\widetilde{M}\gamma_0\,. \tag{4.8}$$

The spacetime split of a bivector is given by

$$F = \mathbf{E} + i\mathbf{B} \tag{4.9}$$

where

$$\mathbf{E} = \tfrac{1}{2}(F + F^\dagger), \quad i\mathbf{B} = \tfrac{1}{2}(F - F^\dagger)\,. \tag{4.10}$$

The split of an electromagnetic field into electric and magnetic parts is precisely of this type.

A multivector $M$ is said to be *even* if it has the expanded form

$$M = \alpha + F + i\beta\,. \tag{4.11}$$

Note that if the split (4.9) is inserted in (4.11), it takes the form (2.7) of a generic multivector in $\mathcal{G}_3$. Thus $\mathcal{G}_3$ is the subalgebra of even multivectors in STA with a given spacetime split.

Any *Lorentz rotation* of the orthonormal frame $\{\gamma_\mu\}$ into another orthonormal frame $\{e_\mu\}$ can be expressed in the form

$$e_\mu = R\,\gamma_\mu \widetilde{R}\,, \tag{4.12}$$

where $R$ is an even multivector with the normalization

$$R\widetilde{R} = 1\,. \tag{4.13}$$

An even multivector $R$ representing a rotation in this way is called a *spinor* or, with the normalization (4.13), a *rotor*.

## 5. SPINOR PARTICLE MECHANICS

Let $x = x(\tau)$ be the timelike history of a particle parameterized by its proper time. The particle's velocity is $v = \dot{x}$, where now the overdot indicates differentiation with respect to proper time, so that

$$v^2 = 1\,. \tag{5.1}$$

Because of this constraint, $v = v(\tau)$ can only rotate as it "moves" along the history. Therefore, it can be expressed as a Lorentz rotation

$$v = R\,\gamma_0 \widetilde{R} \tag{5.2}$$

determined by a spinor $R = R(\tau)$. By differentiating (4.13) it can be proved that

$$\Omega = 2\dot{R}\widetilde{R} \tag{5.3}$$

is necessarily bivector-valued. Differentiating (5.2) we find that

$$\dot{v} = \Omega \cdot v\,, \tag{5.4}$$

where $\Omega \cdot v = \frac{1}{2}(\Omega v - v\Omega)$. This shows that an equation of motion for $v$ necessarily has the form (5.4) with a specified functional dependence for $\Omega$. Alternatively, for specified $\Omega$, (5.3) can be regarded as an equation of motion

$$\dot{R} = \tfrac{1}{2}\Omega R \tag{5.5}$$

for the spinor $R$, from which $v = v(\tau)$ can be obtained by (5.2). Aside from [5], where it was introduced and studied, the spinor equation (5.5) has been largely overlooked in the literature, probably because it cannot be written down, let alone solved, without STA. Nevertheless, it has considerable advantages over (5.4), which is ordinarily employed in a tensor form.

One major advantage of the spinor equation (5.5) is that it determines not merely $v$, but, with $v = e_0$ in (4.12), it determines the precession of an orthornormal frame comoving with the velocity along the history. If the comoving frame is identified with the principal axes of a small rigid body, integrating both translational and rotational motion of the body in a single equation.

A second major advantage of the spinor equation (5.5) is that it can be derived as a classical limit of the Dirac equation [6], describing a classical charged particle with spin and the quantum mechanically correct gyromagnetic ratio $g = 2$. Indeed, for some cases the solution of (5.5) is identical to the solution of the Dirac equation. Thus, through (5.5) the classical limit is more simple, direct and complete than in other approaches.

For a test particle with charge to mass ratio $\epsilon$ moving in an electromagnetic field $F$,

$$\Omega = \epsilon F \tag{5.6}$$

and (5.4) becomes the Lorentz force equation

$$\dot{v} = \epsilon F \cdot v\,. \tag{5.7}$$

Exact solutions of the corresponding spinor equation have been found and studied in detail for three cases [5]: a constant field, a plane wave and a Coulomb field.

For constant $F$, the solution of the spinor equation (5.5) is a simple exponential function, though a second integration to get the particle's history $x = x(\tau)$ requires a neat trick if complexities are to be avoided in the general case [5]. When $F$ is a plane wave field, the spinor solution to (5.5) is identical to the Volkov solution of the Dirac equation [6], except for a phase factor determined by the classical Hamilton-Jacobi equation. This simplifies the form and interpretation of the Volkov solution significantly. The spinor solution to (5.5) for a Coulomb field has not, to my knowledge, appeared elsewhere in the literature. As it has some remarkable features which ought to be widely known, I describe it here in some detail with an eye to applications and generalization.

We consider a massive point source, so it can be regarded as at rest in the inertial system defined by $\gamma_0$, and we place it at the origin $\mathbf{r} = 0$. We suppose that its field is derivable from a central potential $V = V(r)$, so

$$\epsilon F = \triangledown \wedge (V\gamma_0) = -\gamma_0 \wedge \triangledown V = -\nabla V = -\hat{\mathbf{r}}\,\partial_r V\,, \tag{5.8}$$

where $\triangledown = \partial_x$ is the derivative with respect to the spacetime point $x$, which admits the spacetime split

$$\gamma_0 \triangledown = \gamma_0 \cdot \triangledown + \gamma_0 \wedge \triangledown = \partial_t + \nabla\,, \tag{5.9}$$

with $\nabla = \partial_{\mathbf{r}}$, so $\nabla V = (\nabla r)\partial_r V$.

Constants of the motion derive from the fact that $\gamma_0$ is a preferred direction. From (4.2a), a spacetime split of $v$ is

$$v\gamma_0 = v \cdot \gamma_0 + v \wedge \gamma_0 = \dot{t} + \dot{\mathbf{r}}\,. \tag{5.10}$$

From a spacetime split of the equation of motion (5.7) we obtain

$$\ddot{t} = \gamma_0 \cdot \dot{v} = \epsilon\gamma_0 \cdot F \cdot v = -\dot{\mathbf{r}} \boldsymbol{\cdot} \nabla V = \dot{V} - \dot{t}\partial_t V\,. \tag{5.11}$$

Therefore, for a static potential ($\partial_t V = 0$), the *energy*

$$W \equiv v \cdot \gamma_0 + V = \dot{t} + V \tag{5.12}$$

is a constant of the motion. The other part of the split of (5.7) gives us

$$\frac{d}{d\tau}(v \wedge \gamma_0) = \dot{v} \wedge \gamma_0 = \epsilon(F \cdot v) \wedge \gamma_0 = \epsilon F v \cdot \gamma_0\,, \tag{5.13}$$

or

$$\ddot{\mathbf{r}} = -\dot{t}\,\hat{\mathbf{r}}\,\partial_r V\,. \tag{5.14}$$

This implies conservation of the *angular momentum*

$$\mathbf{L} = \mathbf{r} \wedge \dot{\mathbf{r}} = \ell\,\mathbf{i}\,, \tag{5.15}$$

which differs from the Newtonian expression (3.2) only by replacing the coordinate time derivative with a proper time derivative. With that replacement, many of the equations in Section 3 apply here as well; in particular, the eqns. (3.3) to (3.7) and (3.9) to (3.11) with $v \equiv |\,\dot{\mathbf{r}}\,|$.

We are now well prepared to complete our analysis of the Coulomb potential and compare results with the nonrelativistic treatment in Section 3. Steve Gull [7] has recently shown that the orbital motion can be described by the KS-equation, just as in the nonrelativistic case. Introducing the KS parameter $s$ by (3.18) as before, for the Coulomb potential $V = -k/r$, (5.14) integrates immediately to

$$W\tau = t - ks\,. \tag{5.16}$$

Since $\partial_r V = -V/r$ for the Coulomb potential, from (5.16) and (5.14) we obtain

$$\ddot{\mathbf{r}}\mathbf{r} = \dot{t}V = \tfrac{1}{2}(W^2 - \dot{t}^2 - V^2)\,.$$

Using (5.12) to eliminate $\dot{t}^2$, this can be put in the form

$$\ddot{\mathbf{r}}\mathbf{r} + \tfrac{1}{2}(\dot{\mathbf{r}}^2 - V^2) = \tfrac{1}{2}(W^2 - 1)\,. \tag{5.17}$$

By virtue of (3.5), for the Coulomb potential we can write

$$\dot{\mathbf{r}}^2 - V^2 = \dot{r}^2 + \frac{(\ell^2 - k^2)}{r^2}\,. \tag{5.18}$$

Thus, by shifting the effective angular momentum we can absorb the $V^2$ term into $\dot{\mathbf{r}}^2$ and insert (5.17) into (3.9) to get the *relativistic K-S equation*

$$\frac{d^2U}{ds^2} = \tfrac{1}{4}(W^2 - 1)U\,. \tag{5.19}$$

However, the spinor $U$ is no longer related to the position vector $\mathbf{r}$ by (2.11). Instead, we have

$$U\boldsymbol{\sigma}_1\widetilde{U} = r\boldsymbol{\sigma}_1 e^{\mathbf{i}\alpha\theta} = \mathbf{r}e^{\mathbf{i}(\alpha-1)\theta} \tag{5.20}$$

where, for $\ell > k$,

$$\alpha = \left(\frac{\ell^2 - k^2}{\ell^2}\right)^{\frac{1}{2}}\,. \tag{5.21}$$

This shows explicitly that $U$ specifies the particle position in a precessing reference frame.

As in the nonrelativistic case, the bound state ($W^2 > 1$) solutions to (5.19) are given by (3.22), although the frequency is given by $\omega^2 = |\,W^2 - 1\,|$ instead of (3.23). The proper time $\tau$ is related to the K-S parameter $s$ by (3.25), and the two of them are related to the coordinate time $t$ by (5.16). In the nonrotating reference frame of the stationary "nucleus," the orbit is seen as a precessing ellipse. In the Newtonian case, only orbits with $\ell = 0$ pass through the origin.

For sufficiently small angular momentum ($\ell \leq k$) we see from (5.18) that the "centrifugal potential" is insufficient to absorb the "relativistic correction" and prevent even hyperbolic orbits from passing through $r = 0$. As Gull has pointed out, the critical value $\ell = k$ occurs in a Bohr atom with atomic number $Z = 137$, and a similar "crisis" occurs in quantum mechanical Coulomb solutions of the Dirac equation with the same atomic number. Of course, in the real physical situation there are other factors to prevent the particle from reaching the origin, so this crisis should not be taken too seriously. Nevertheless, a study of the critical solutions is of some interest. The distinctive features of these solutions are most evident in the general solution of the spinor equation (5.5), to which we now turn.

Angular momentum conservation implies that the orbit lies in the plane of $\hat{\mathbf{L}} = \mathbf{i}$. Therefore, as in (2.11), we can refer the position vector with respect to a fixed direction $\boldsymbol{\sigma}_1$ in the plane by writing

$$\hat{\mathbf{r}} = U_\theta \boldsymbol{\sigma}_1 \widetilde{U}_\theta = U_\theta^2 \boldsymbol{\sigma}_1 = e^{-\mathbf{i}\theta} \boldsymbol{\sigma}_1 \tag{5.22}$$

where

$$U_\theta = e^{-\frac{1}{2}\mathbf{i}\theta}\,, \tag{5.23}$$

with

$$\mathbf{i} = i\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\,. \tag{5.24}$$

Inserting this into (3.3), we obtain the standard relation

$$\ell = r^2\dot{\theta}\,. \tag{5.25}$$

For a Coulomb potential $V = -k/r$ the spinor equation (5.5) becomes

$$\dot{R} = -\frac{k\hat{\mathbf{r}}}{2r^2}\,R\,. \tag{5.26}$$

Using (5.22) to change variables, this simplifies to

$$\frac{dR}{d\theta} = -\frac{\kappa}{2}\,\hat{\mathbf{r}}R \tag{5.27}$$

where $\kappa = k/\ell$. This has the general solution

$$R = U_\theta S_\theta L_0\,, \tag{5.28}$$

where $U_\theta$ is defined by (5.23),

$$S_\theta = e^{-\frac{1}{2}A\theta} \tag{5.29}$$

with

$$A = \kappa\boldsymbol{\sigma}_1 - \mathbf{i} = \boldsymbol{\sigma}_1(\kappa + \boldsymbol{\sigma}_2)\,, \tag{5.30}$$

and

$$v_0 = L_0\gamma_0\widetilde{L}_0 = L_0^2\gamma_0 \tag{5.31}$$

is the velocity at $\theta = 0$. It is convenient to align $\boldsymbol{\sigma}_1$ with a pericenter, so that

$$v_0\gamma_0 = L_0^2 = (W - V_0) + [(W - V_0)^2 - 1]^{\frac{1}{2}}\,\boldsymbol{\sigma}_2 \tag{5.32}$$

with $V_0 = -k/r_0$.

The solution (5.28) is remarkable for its simplicity and structure. It reveals that the bivector $A$ is a constant of the motion along with the bivector $\mathbf{L}$. We can regard $A$ as a generalization of the eccentricity vector (3.8), which is no longer constant in the relativistic case, where the pericenter precesses. The quantity $A^2 = \kappa^2 - 1$ distinguishes three types of motion: bounded velocity for $\kappa < 1$, unbounded velocity for $\kappa > 1$ and a "critical" case for $\kappa = 1$. In accord with (5.21) we write

$$\alpha = |\,A\,| = |\,\kappa^2 - 1\,|^{\frac{1}{2}}\,. \tag{5.33}$$

For bounded velocity we can express $A$ as a boost of $\mathbf{i}$ by a constant rotor $K$:

$$A = -\alpha K\mathbf{i}\widetilde{K} = -\alpha K^2\mathbf{i}\,, \tag{5.34}$$

whence

$$K^2 = \alpha^{-1}A\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = \frac{1 + \kappa\boldsymbol{\sigma}_2}{(1 + \kappa^2)^{\frac{1}{2}}}\,. \tag{5.35}$$

Therefore, we can express (5.29) as a composite of constant boost with a rotation:

$$S_\theta = Ke^{-\frac{1}{2}\mathbf{i}\alpha\theta}\widetilde{K}\,. \tag{5.36}$$

For unbounded velocity we can write

$$A = \alpha K\boldsymbol{\sigma}_1\widetilde{K} = \alpha K^2\boldsymbol{\sigma}_1\,, \tag{5.37}$$

so

$$K^2 = \alpha^{-1}A\boldsymbol{\sigma}_1 = \frac{\kappa + \boldsymbol{\sigma}_2}{(\kappa^2 - 1)^{\frac{1}{2}}} \tag{5.38}$$

and

$$S_\theta = Ke^{-\frac{1}{2}\boldsymbol{\sigma}_2\alpha\theta}\widetilde{K}\,. \tag{5.39}$$

These results exhibit the solution (5.28) as a composite of boosts and rotations in fixed timelike and spacelike planes. In the bounded case (5.36) the spinor $S_\theta$ describes a continuous rotation, while in the unbounded case (5.39) it describes a continuously increasing boost.

With these results any question about motion in a Coulomb field can be answered with algebraic calculations, just as in the Newtonian case. For example, the relativistic deflection formula for Coulomb scattering can be derived (in much the same way as in Section 3) by using (5.2), (5.28) and (5.32) to evaluate (3.11).

To generalize the Coulomb solution (5.28) to an arbitrary central field, note first that, since $\mathbf{L} = \ell\mathbf{i}$ is conserved, the algebraic generators of motion are limited to generators of rotations and boosts in the $\mathbf{i}$-plane, for example, $\mathbf{i}$, $\boldsymbol{\sigma}_2$ and their *commutator product* $\mathbf{i} \times \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1$. However, our Coulomb experience suggests that it may be better to use the generators $\mathbf{i}$ and $A$, which have the commutator product $\mathbf{i} \times A = \kappa\boldsymbol{\sigma}_1$. This leads us to construct the trial solution

$$R = e^{-\frac{1}{2}\mathbf{i}\varphi} e^{-\frac{1}{2}A\mu} e^{-\frac{1}{2}\mathbf{i}\lambda} L_0\,, \tag{5.40}$$

where $\varphi$, $\mu$ and $\lambda$ are scalar functions of proper time to be found from the given potential function $V(r)$ by substituting (5.40) into the equation of motion

$$\dot{R} = -\tfrac{1}{2}\boldsymbol{\sigma}_2 e^{\mathbf{i}\theta}(\partial_r V)R\,. \tag{5.41}$$

Beyond this, there is much to be done generalizing this approach to incorporate the effects of perturbing forces, especially external magnetic fields and plane wave fields.

## 6. GRAVITATIONAL MOTION AND PRECESSION

The spinor equation of motion (5.5) has been used to describe the translational and rotational motion of a small rigid body in a gravitation field according General Relativity [8]. Recently, however, Lasenby, Doran and Gull [9] have proposed a new gauge theory of gravity that employs geometric algebra in an essential way, and appears to be a significant improvement on General Relativity. Here we outline its application to motion in the static gravitational field of a fixed point mass and see how it compares to the Coulomb problem in the preceding section, recalling that the two motions are essentially identical according to Newtonian Theory.

In the gauge theory, the spinor equation for motion in a gravitational field has the same form as (5.5):

$$\dot{R} = \tfrac{1}{2}\Omega(v)R\,, \tag{6.1}$$

except that the bivector gauge field $\Omega(v) = \Omega(v, x)$ is a linear function of $v = R\gamma_0\widetilde{R}$. Moreover, $v$ itself is a linear function of the velocity $\dot{x}$ given by

$$v = \underline{h}^{-1}(\dot{x})\,, \tag{6.2}$$

where the gauge tensor field $\underline{h}^{-1}$ plays the role of gravitational potential. Let us refer to $v$ as the *celerity* of the particle to distinguish it from the *velocity* $\dot{x}$.

As before, let the constant vector $\gamma_0$ characterize the reference system in which the point source is "at rest," and let $\hat{\mathbf{r}} = e_r \wedge \gamma_0 = e_r\gamma_0$ represent the radial direction from its position at the origin. As explained in [8], the gravitational potential is most naturally represented in the "Newtonian gauge," where, in the notation of [9], it takes the form

$$v = \underline{h}^{-1}(\dot{x}) = \dot{x} - g_2(\dot{x}\cdot\gamma_0)e_r\,, \tag{6.3}$$

with, for a source of mass $M$,

$$g_2 \equiv -\sqrt{\frac{2M}{r}}\,, \tag{6.4}$$

with $r = |\,\mathbf{r}\,| = |\,x \wedge \gamma_0\,|$. In this case

$$\Omega(v) = \bar{\nabla} \wedge \bar{h}^{-1}(\grave{v}), \tag{6.5}$$

where the accent over $v$ serves to indicate that $v$ is not differentiated by

$$\bar{\nabla} \equiv \bar{h}(\nabla) = \nabla + g_2\gamma_0\, e_r \cdot \nabla, \tag{6.6}$$

where $\bar{h}$ is the adjoint of $\underline{h}$, and, as in (5.9), $\nabla$ is the derivative with respect to $x$. Consequently,

$$\bar{h}^{-1}(v) = v - g_2\gamma_0 e_r \cdot v \tag{6.7}$$

and

$$\Omega(v) = \gamma_0 \wedge \bar{\nabla}\,(g_2 e_r \cdot \grave{v}) = \gamma_0 \wedge \nabla(g_2 e_r \cdot \grave{v}). \tag{6.8}$$

Since $\gamma_0$ is a preferred direction of the gravitational field, a spacetime split is in order to take advantage of this symmetry. Thus, from (6.3)

$$v \cdot \gamma_0 = \dot{x} \cdot \gamma_0 = \dot{t}, \tag{6.9}$$

and

$$\mathbf{v} \equiv v \wedge \gamma_0 = \dot{\mathbf{r}} - g_2\dot{t}\hat{\mathbf{r}}. \tag{6.10}$$

Since $e_r \cdot v = -\hat{\mathbf{r}} \cdot \mathbf{v}$, we can put (6.8) in the form

$$\Omega(v) = -\nabla\Big(g_2 \frac{\mathbf{r} \cdot \grave{\mathbf{v}}}{r}\Big) = -\frac{g_2}{r}\,\big[\,\mathbf{v} - \tfrac{3}{2}(\mathbf{v} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\,\big], \tag{6.11}$$

where differentiation (without coordinates) was performed by using the vector derivatives $\nabla r = \hat{\mathbf{r}}$ and $\nabla \mathbf{r} \cdot \grave{\mathbf{v}} = \mathbf{v}$.

Now, before we can solve the spinor equation of motion (6.2), we need to relate $\Omega(v)$ in (6.11) to parameters of the orbit. As before, the necessary relations come from a spacetime split of the equation (5.4) for $\dot{v}$. The time component of (5.4) yields a "generalized energy" constant of motion

$$W \equiv v \cdot \underline{h}^{-1}(\gamma_0) = \dot{t} + g_2\mathbf{v} \cdot \hat{\mathbf{r}} = g_2\dot{r} + (1 + g_2^2)\dot{t}. \tag{6.12}$$

The space component yields

$$\dot{\mathbf{v}} = \Omega(v)\,(v \cdot \gamma_0) = -\frac{\dot{t}g_2}{r}\big[\,\mathbf{v} - \tfrac{3}{2}(\mathbf{v} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\,\big]. \tag{6.13}$$

As before, this implies the conserved angular momentum

$$\mathbf{L} \equiv \mathbf{r} \wedge \mathbf{v} = \mathbf{r} \wedge \dot{\mathbf{r}} = \ell\,\mathbf{i}. \tag{6.14}$$

Using (3.4) and (6.12) we can write

$$\mathbf{v} - \tfrac{3}{2}\hat{\mathbf{r}}\hat{\mathbf{r}} \cdot \mathbf{v} = -\hat{\mathbf{r}}\big[\,\frac{\mathbf{L}}{r} - \tfrac{1}{2}\hat{\mathbf{r}} \cdot \mathbf{v}\,\big] = -\hat{\mathbf{r}}\,[\,\mathbf{i}r\dot{\theta} + \tfrac{1}{2}(\dot{t} - W)\,]. \tag{6.15}$$

Putting this into (6.11) and using (5.22) we get the spinor equation of motion

$$2\dot{R}\widetilde{R} = \frac{g_2}{r}\,[\,\tfrac{1}{2}(\dot{t} - W) - \mathbf{i}\dot{\theta}\,]\,e^{-\mathbf{i}\theta}\boldsymbol{\sigma}_1. \tag{6.16}$$

This is now of such a form that we can expect to solve it with the same trial solution (5.40) suggested for the electromagnetic central force case. Answers to questions about scattering, orbital motion and precession can be found in the same way as before—only the functional forms are different.

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