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Projective Geometry with Clifford Algebra*

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Abstract. Projective geometry is formulated in the language of geometric algebra, a unified mathematical language based on Clifford algebra. This closes the gap between algebraic and synthetic approaches to projective geometry and facilitates connections with the rest of mathematics.

1. Introduction

Despite its richness and influence in the nineteenth century, projective geometry has not been fully integrated into modern mathematics. The reason for this unfortunate state of affairs is to be found in certain incompatibilities of method. The ordinary synthetic and coordinate-based methods of projective geometry do not meld well with the popular mathematical formalisms of today. However, the foundation for a more efficient method had already been laid down in the nineteenth century by Hermann Grassmann (1809-1877), though, to this day, that fact has been appreciated by only a few mathematicians. This point has been argued forcefully by Gian-Carlo Rota and his coworkers [1,2,16]. They claim that Grassmann's progressive and regressive products are cornerstones of an ideal calculus for stating and proving theorems in invariant theory as well as projective geometry. From that perspective they launch a telling critique of contemporary mathematical formalism. Their main point is that the formalism should be modified to accommodate Grassmann's regressive product, and they offer specific proposals for doing so. We think that is a step in the right direction, but it does not go far enough. In this article, we aim to show that there is a deeper modification with even greater advantages.

We see the problem of integrating projective geometry with the rest of mathematics as part of a broad program to optimize the design of mathematical systems [8,10]. Accordingly, we seek an efficient formulation of projective geometry within a coherent mathematical system which provides equally efficient formulations for the full range of geometric concepts. A *geometric calculus* with these characteristics has been under development for some time. Detailed applications of geometric calculus have been worked out for a large portion of mathematics [12] and nearly the whole of physics [7-9]. It seems safe to claim that no single alternative system has such a broad range of applications. Thus, by expressing the ideas and results of projective geometry in the language of geometric calculus, we can make

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them readily available for applications to many other fields. We hope this will overcome the serious 'language barrier' which has retarded the diffusion of projective geometry in recent times.

Clifford algebra is the mathematical backbone of geometric calculus. The mistaken belief that Clifford algebra applies only to metric spaces has severely retarded recognition of its general utility. We hope to dispel that misconception once and for all by demonstrating the utility of Clifford algebra for expressing the nonmetrical concepts of projective geometry. This strengthens the claim, already well grounded, that Clifford algebra should be regarded as a universal geometric algebra. To emphasize its geometric significance, we will henceforth refer to Clifford algebra by the descriptive name *geometric algebra*, as Clifford himself originally suggested.

In geometric algebra there is a single basic kind of multiplication called the *geometric product*. In terms of this single product, a great variety (if not all) of the important algebraic products in mathematics can be simply defined (see [8,12]). This provides a powerful approach to a unified theory of algebraic (and geometric) structures, for it reduces similarities in different algebraic systems to a common body of relations, definitions, and theorems. We think the decision to follow that approach is a fundamental issue in the design of mathematical systems.

In this article, we define Grassmann's progressive and regressive products in terms of the geometric product and derive their properties therefrom. These properties include a system of identities which have been derived and discussed by many authors dating back to Grassmann. However, the approach from geometric algebra is sufficiently different to justify reworking the subject once more. The central geometrical idea is that, with suitable geometric interpretations, the identities provide straightforward proofs of theorems in projective geometry. This idea also originated with Grassmann, though it has been much elaborated since.

Since Grassmann's progressive and regressive products can be (and have been) directly defined and applied to projective geometry, the suggestion that they be regarded as subsidiary to the geometric product requires a thorough justification. The most important reason has already been mentioned and will be elaborated on in subsequent discussions, namely, the geometric product provides connections to other algebraic and geometric ideas. Specific connections to affine and metric geometry will be discussed in a subsequent article [11]. Within the domain of applications to projective geometry alone, however, we believe that the geometric product clarifies the role of duality and enhances the fluidity of expressions. Moreover, some important relations in projective geometry are not readily expressed in terms of progressive and regressive products without an abuse of notation. A prime example is the cross-ratio discussed in [11].

Geometric algebra will lead us to conclude that Grassmann's inner product is more fundamental than his closely related regressive product, so our use of the latter will be limited. However, we need not make hard choices between them, since translation from one product to the other is so easy. The same can be said about their relation to the geometric product. Indeed, the progressive and inner products together are essentially equivalent to the geometric product, as should be clear from the way we obtain them by 'taking the geometric product apart.' The crucial converse step of joining them into a single product was finally taken by Grassmann in one of his last published articles [6] (see [8] for comments). Ironically, that seminal article was dismissed as without interest by his biographer Engel [3] and ignored by everyone else since.

The main purpose of this article is to set the stage for a complete treatment of projective geometry with the language and techniques of geometric algebra. This requires first that we establish the necessary definitions, notations and geometric interpretation. Sections 2 and 3 are devoted to this task. Next, we show how the algebra is used to formulate and prove a representative set of important theorems in projective geometry. As our objective is to rebottle the old wine of projective geometry, we do not have new results to report. The originality of this article lies solely in the method. Nevertheless, we know that much of the old material we discuss will be unfamiliar to many readers, so we hope also to help them reclaim the past. Finally, in the appendix, we present a guide for translating the rich store of ideas and theorems in the literature into the language of geometric algebra. The serious student will want to delve into the literature to see what treasures have yet to be reclaimed.

2. Geometric Algebra

In this section we discuss features of geometric algebra needed for our treatment of projective geometry. We follow the extensive treatment of geometric algebra in [12], so we can omit many details covered there. In particular, we omit proofs of the basic identities except where we wish to emphasize important methodological points. The fact that our formulation of geometric algebra is completely coordinate-free is especially important, because it makes possible a completely coordinate-free treatment of projective geometry.

2.1. BASIC DEFTNITIONS

Let $\mathcal{V}_n$, be an n-dimensional vector space over the reals. Throughout this article, lowercase letters $a, b, c, \ldots$ denote vectors and lowercase Greek letters denote (real) scalars. The *geometric algebra* $\mathcal{G}_n = \mathcal{G}(\mathcal{V}_n)$ is generated from $\mathcal{V}_n$ by defining the *geometric product* ab with the following properties holding for all vectors

$$a(bc) = (ab)c, \tag{2.1a}$$

$$a(b + c) = ab + ac, \tag{2.1b}$$

$$(b + c)a = ba + ca, \tag{2.1c}$$

$$a\lambda = \lambda a \tag{2.1d}$$

$$a^2 = \pm|\, a\,|^2, \tag{2.1e}$$

where $|\, a\,|$ is a positive scalar associated with a. Axiom (2.1e) is called the *contraction rule*. The vector a is said to have *positive (or negative) signature* when the sign in (2.1e) is specified as positive (or negative), and a is said to be a *null vector* if $|\, a\,| = 0$ when $a \neq 0$.

To avoid trivialization, the above axioms must be supplemented by an axiom ensuring that the product ab of nonzero vectors vanishes only if the vectors are collinear and null. With the additional assumption that $\mathcal{G}_n$ is not generated by any proper subspace of $\mathcal{G}_n$, it can be proved that with the geometric product $\mathcal{G}_n$ generates exactly 2^n linearly independent elements. Thus, $\mathcal{G}_n = \mathcal{G}(\mathcal{V}_n)$ is a 2^n-dimensional algebra. Actually, there are different types of geometric algebra distinguished by specifications on the contraction rule. If all vectors are assumed to be null, then $\mathcal{G}_n$ is exactly the Grassmann algebra of $\mathcal{G}_n$. However, as shown below, the Grassmann algebra is included in every type of $\mathcal{G}_n$. Now, let p and q be, respectively, the dimension of maximal subspaces of vectors with positive and negative signature. The different types of geometric algebra distinguished by the different signatures can be distinguished by writing $\mathcal{G}_n = \mathcal{G}(p, q)$. If $p+q = n$, the algebra $\mathcal{G}_n$ and its contraction rule are said to be *nondegenerate with signature* (p, q). We deal only with nondegenerate algebras, because all other cases are included therein.

A geometric algebra is said to be *Euclidean* if its signature is $(n, 0)$ or *anti-Euclidean* if its signature is $(0, n)$. For the purpose of projective geometry it is convenient to adopt the Euclidean signature, though all algebraic relations that arise are independent of signature with the exception of a few which degenerate on null vectors. In this paper we will ignore signature except for occasional remarks in places where it can make a difference. In the companion paper [11], signature becomes important when projective geometry is related to metrical geometry.

From the *geometric product* ab, two new kinds of product can be defined by decomposing it into symmetric and antisymmetric parts. Thus,

$$ab = a \cdot b + a \wedge b\,, \tag{2.2}$$

where the *inner product* $a \cdot b$ is defined by

$$a \cdot b = \tfrac{1}{2}(ab + ba) \tag{2.3a}$$

and the *outer product* $a \wedge b$ is defined by

$$a \wedge b = \tfrac{1}{2}(ab - ba) \tag{2.3b}$$

In consequence of the contraction axiom, the inner product is scalar-valued. The outer product of any number of vectors $a_1, a_2, \ldots, a_k$ can be defined as the completely antisymmetric part of their geometric product and denoted by

$$\langle\, a_1 a_2 \cdots a_k \,\rangle_k = a_1 \wedge a_2 \wedge \cdots \wedge a_k\,. \tag{2.4}$$

This can be identified with Grassmann's progressive product, though we prefer the alternative term 'outer product.'

Any element of $\mathcal{G}_n$ which can be generated by the outer product of k vectors, as expressed in (2.4), is called a *k-blade or a blade of step (or grade) k*. Any linear combination of k-blades is called a *k-vector*. The k-fold outer product (2.4) vanishes if and only if the k-vectors are linearly independent. Therefore, $\mathcal{G}_n$ contains nonzero blades of maximum step n. These n-blades are called *pseudoscalars* of $\mathcal{V}_n$ or of $\mathcal{G}_n$. A generic element of $\mathcal{G}_n$ is called a *multivector*. Every multivector M in $\mathcal{G}_n$ can be written in the expanded form

$$M = \sum_{k=0}^{n} \langle\, M \,\rangle_k\,, \tag{2.5}$$

where $\langle M \rangle_k$ denotes the *k-vector part* of M. As in (2.5), it is sometimes convenient to refer to scalars as 0-vectors; but we employ the term blade only for step $k \geq 1$. Since the scalar part is of special interest so often, we often write $\langle M \rangle$ for $\langle M \rangle_0$.

The main antiautomorphism of $\mathcal{G}_n$ is called *reversion*, and $M^\dagger$ denotes the *reverse* of M. From the antisymmetry of the outer product (2.4), it follows that

$$\langle M^\dagger \rangle_k = \langle M \rangle_k^\dagger = (-1)^{k(k-1)/2} \langle M \rangle_k \,. \tag{2.6}$$

Every blade A in $\mathcal{G}_n$ has the property that its square is a scalar, that is,

$$A^2 = \langle A^2 \rangle \,. \tag{2.7}$$

We say that a nonzero blade A is null if $A^2 = 0$.

The algebra $\mathcal{G}_n$ is nondegenerate if and only if the square I^2 of any nonvanishing pseudoscalar I is a nonvanishing scalar. Throughout this article, the symbol I denotes *unit pseudoscalar*. The sign of I^2 depends on the *signature* of $\mathcal{G}_n = \mathcal{G}(p, q)$ according to

$$II^\dagger = (-1)^q |\, I \,|^2 \,, \tag{2.8}$$

where $|\, I^2 \,| = |\, I \,|^2$ defines the 'magnitude of I.' The choice $|\, I \,| = 1$ fixes a scale for the pseudoscalars, but we need not make that choice because we will be concerned only with 'relative magnitudes.' The magnitude of a pseudoscalar P relative to I will be called the *bracket* of P and defined by

$$[\, P \,] = PI^{-1} \,, \tag{2.9}$$

so $[\, I \,] = II^{-1} = 1$. It is important to note that the sign of the bracket is independent of signature. For the bracket determined by n vectors, we write

$$[\, a_1 a_2 \cdots a_n \,] \equiv [\, a_1 \wedge a_2 \cdots \wedge a_n \,] = (a_1 \wedge a_2 \cdots \wedge a_n) \cdot I^{-1} \,. \tag{2.10}$$

This can be taken as the definition of a determinant, and therefrom the whole theory of determinants can be developed efficiently with geometric algebra. Details are given in [8].

2.2. IDENTITIES

We will be concerned primarily with the algebraic relations among blades, since blades have a direct geometric interpretation in projective geometry. To express and derive these relations, we need a few basic properties of inner and outer products. For blades A and B of step r and s, respectively,

$$A \wedge B = \langle AB \rangle_{r+s} = (-1)^{rs} B \wedge A \,; \tag{2.11}$$

$$A \cdot B = \langle AB \rangle_{|\, r-s \,|} = (-1)^{r(s-r)} B \cdot A \qquad \text{for} \quad s \geq r \,. \tag{2.12}$$

These can be taken as definitions of inner and outer products in terms of the geometric product AB, generalizing (2.3a,b).

The geometric product of a vector with a blade B of step s can now be decomposed into a 'step down' part

$$a \cdot B = \langle\, aB \,\rangle_{s-1} = (-1)^{s-1} B \cdot a \tag{2.13}$$

and a 'step up' part

$$a \wedge B = \langle\, aB \,\rangle_{s+1} = (-1)^{s} B \wedge a\,. \tag{2.14}$$

Thus,

$$aB = a \cdot B + a \wedge B\,. \tag{2.15}$$

This generalizes (2.2).

For $A = a_1 \wedge a_2 \wedge \cdots \wedge a_r$ of step r and $B = b_1 \wedge b_2 \wedge \cdots \wedge b_s$ of step $s \le r$, the inner product can be developed in the expansion

$$\begin{aligned} B \cdot (a_1 \wedge a_2 \wedge \cdots \wedge a_r) &= B \cdot (a_1 \wedge \cdots \wedge a_s)\, a_{s+1} \wedge \cdots \wedge a_r + \\ &\quad + (-1)^{s-1} B \cdot (a_2 \wedge \cdots \wedge a_{s+1})\, a_1 \wedge a_{s+2} \wedge \cdots \wedge a_r + \cdots \\ &= \sum \epsilon(j_1 j_2 \cdots j_r)\, B \cdot (a_{j_1} \wedge a_{j_2} \wedge \cdots \wedge a_{j_s})\, a_{j_{s+1}} \wedge \cdots \wedge a_{j_r} \end{aligned} \tag{2.16}$$

where the sum is over all j's with $j_1 < j_2 < \cdots < j_r$ and the permutation symbol $\epsilon(j_1 j_2 \cdots j_r)$ has the value 1 (or -1) if $(j_1 j_2 \cdots j_n)$ is an even (or odd) permutation of $(1, 2, \ldots, n)$. This expansion can be regarded as a generalization of the Laplace expansion for determinants. The number of nonvanishing terms in this expansion of $B \cdot A$ depends on the particular set of vectors into which A has been factored. We can always find a factoring $A = a_1 \wedge a_2 \wedge \cdots \wedge a_r$ such that

$$B = \alpha(b_1 + a_1) \wedge (b_2 + a_2) \wedge \cdots \wedge (b_s + a_s)\,,$$

where, for $i = 1, 2, \ldots, s$, $a_i \cdot a_j = 0$ for $j = s+1, s+2, \ldots, r$ and $b_i \cdot a_j$ for $j = 1, 2, \ldots, r$. In that case (2.16) reduces to a single term

$$B \cdot (a_1 \wedge a_2 \wedge \cdots \wedge a_r) = \alpha (a_1 \wedge \cdots \wedge a_s)^2\, a_{s+1} \wedge a_{s+2} \wedge \cdots \wedge a_r\,.$$

Thus we may conclude that the inner product of two blades always produces a blade.

The outer product is associative but the inner product is not; rather it obeys the *following rule of the middle factor:* For blades A, B, C with steps r, s, t, respectively,

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C \quad \text{if} \quad r + t \le s\,, \tag{2.17}$$

$$A \cdot (B \cdot C) = (A \wedge B) \cdot C \quad \text{if} \quad r + s \le t\,. \tag{2.18}$$

A number of useful identities can easily be derived as needed by combining this with the expansion (2.16).

2.3 DUALITY

We define the *dual* A of an r-blade $\widetilde{A}$ by

$$\widetilde{A} = AI^{-1} = A \cdot I^{-1} = (-1)^{r(n-r)} I^{-1} A\,. \tag{2.19}$$

(This corrects an important misprint on p. 7 of [8].)

It follows that the dual of an r-blade is an $(n-r)$-blade. In addition, the sum of two $(n-1)$ blades is always an $(n-1)$-blade, hence all $(n-1)$-vectors are $(n-1)$-blades, since the dual of an $(n-1)$-blade is a 1-vector. This can be inferred from the fact that the sum of any number of 1-vectors is always a vector.

From (2.18) and (2.19) we obtain

$$A\cdot(BI) = (A\wedge B)I = (-1)^{s(n-s)}(AI)\cdot B\,, \tag{2.20}$$

or, equivalently,

$$(A\wedge B)^{\sim} = A\cdot\widetilde{B} = (-1)^{s(n-s)}\widetilde{A}\cdot B\,, \tag{2.21}$$

where s is the step of blade B. This expresses the duality of inner and outer products. When $r = n-s$, we can write

$$A\cdot\widetilde{B} = \langle\, A\widetilde{B}\,\rangle = [\,A\wedge B\,] \equiv [\,AB\,]\,, \tag{2.22}$$

showing that the bracket is a special duality relation.

The foregoing helps us to distinguish two different (though intimately related) roles of the inner product in geometric algebra: on the one hand, it may serve a metrical function in the usual way; on the other hand, it serves as a step-lowering operation or *contraction*. The step-lowering property of the inner product expressed by (2.12) is dual to the step-raising property of the outer product expressed by (2.11). Both raising and lowering operations are essential for expressing the nonmetrical relations of projective geometry, as shown in subsequent sections. Whether the inner product performs a metrical or nonmetrical role depends on the context in which it is used. Ordinarily, the inner product $a\cdot b$ in (2.2) is given a metrical interpretation, but the vector b can always be expressed as the dual of a pseudovector B by writing $b = \widetilde{B}$, so by (2.19),

$$a\cdot b = a\cdot\widetilde{B} = [\,a\wedge B\,]\,. \tag{2.23}$$

As noted before, the value of the bracket is nonmetrical in the sense that it is independent of signature. The trick of using the inner product to express nonmetrical relations by combining it with duality has been much used in projective geometry. In fact, it is the essence of the theory of poles and polars.

The inner product is essential to duality, as the definition (2.19) shows. The treatment of duality with geometric algebra differs from the usual approach in the literature dating back to Grassmann by its explicit use of the pseudoscalar. This has the advantage of reducing duality to multiplication by a special element of the algebra rather than introducing it as a special algebraic operation. It simplifies the problem of relating duality operations for intersecting spaces of different dimension.

2.4. THE COMMUTATOR PRODUCT

The *commutator product* of multivectors M and N is defined in terms of the geometric product by

$$M\times N = \tfrac{1}{2}(MN - NM) = -N\times M\,. \tag{2.24}$$

This product is a 'derivation' with respect to the geometric product, that is, for any multivectors L, M, N,

$$L \times (MN) = (L \times M)N + M(L \times N)\,. \tag{2.25}$$

This leads easily to the Jacobi identity,

$$L \times (M \times N) = (L \times M) \times N + M \times (L \times N)\,. \tag{2.26}$$

The relation of the commutator product to inner and outer products is step dependent. Thus, for any vector a,

$$a \times B = a \wedge B\,, \tag{2.27}$$

if B is a blade with odd step, and

$$a \times B = a \cdot B\,, \tag{2.28}$$

if the step of B is even.

The commutator is especially useful for relations and manipulations involving bivectors. For a 2-blade $A = a \wedge b = a \times b$ and any blade B of step $s \geq 2$, the relations (2.27) and (2.28) enable us to put the Jacobi identity in the useful forms

$$\begin{aligned} A \times B = (a \wedge b) \times B &= a \cdot (b \wedge B) - b \cdot (a \wedge B) \\ &= b \wedge (a \cdot B) - a \wedge (b \cdot B)\,. \end{aligned} \tag{2.29}$$

The two forms are related by the identity

$$a \cdot (b \wedge B) = (a \cdot b)B + b \wedge (a \cdot B)\,.$$

Note that (2.28) applies whether the step of B is even or odd. It also shows that the commutator product by a bivector is a step-preserving operator. The geometric product now admits the decomposition

$$AB = A \cdot B + A \times B + A \wedge B\,, \tag{2.30}$$

where the three terms on the right have steps $s-2$, s, and $s+2$, respectively. Note that this decomposition into terms of homogeneous step cannot be expressed in terms of inner and outer products alone without decomposing A into vectors. The three different products on the right side of (2.29) express three different geometrical relations between A and B.

For three 2-blades A, B, C we can prove

$$(A \times B) \cdot C = \langle ABC \rangle = \langle CAB \rangle = (C \times A) \cdot B \tag{2.31}$$

from (2.29) and $\langle MN \rangle = \langle NM \rangle$. Writing $C = c \wedge c'$ and employing (2.29) in (2.31), we derive the identity

$$\langle ABC \rangle = \langle AB(c \wedge c') \rangle = (A \wedge c) \cdot (B \wedge c') - (A \wedge c') \cdot (B \wedge c)\,. \tag{2.32}$$

In the subsequent discussion, it will be seen that the commutator product is not essential to our treatment of projective geometry. Nevertheless, we employ the commutator product in some places because we believe it gives insight into the structure of algebraic relations. Moreover, it is important for establishing relations to other mathematical structures. There are good reasons, for example, for basing the theory of Lie algebras on the commutator product in Geometric Algebra (see [8] and remarks in [9]).

3. The Algebra of Incidence

The elements of $\mathcal{G}_n$ can be assigned a variety of different geometrical interpretations appropriate for different applications. In this section we represent the primitive elements and relations of projective geometry by elements and algebraic operations in geometric algebra. Then we can use geometric algebra to prove the theorems of projective geometry.

3.1. PROGRESSIVE AND REGRESSIVE PRODUCTS

The blades in $\mathcal{G}_n$ determine an 'algebra of subspaces' for $\mathcal{V}_n$ discussed at length in Section 1-2 of [8]. We first review some of the main ideas in that discussion as groundwork for 'projective interpretations.' To every r-dimensional subspace $\mathcal{V}_r$ in $\mathcal{V}_n$ there corresponds an r-blade $A = \langle A \rangle_r$ such that $\mathcal{V}_r$ is the solution set of the equation

$$x \wedge A = 0\,. \tag{3.1}$$

According to this equation, A is unique to within a scale factor. The vector space generates a geometric algebra $\mathcal{G}_r$ in which A is a pseudoscalar. Accordingly, we may refer to A as the *pseudoscalar* or the *blade* of $\mathcal{V}_r$. Conversely, we refer to the vector space $\mathcal{V}(\mathcal{A})$ determined by the blade A as the *support* of A. Since each subspace is uniquely determined by its blade, the lattice algebra of subspaces can be represented by an 'algebra of blades' in $\mathcal{G}_n$ To accomplish that we need only introduce an appropriate system of definitions; the geometric algebra will take care of the rest.

For given blades A and B, if there exists a scalar or blade C such that

$$A = BC = B \wedge C\,. \tag{3.2}$$

we say that A is a *dividend* of B and B is a *divisor (or factor)* of A. As to the geometric interpretation of factoring, it should be noted that B is a factor of A if and only if the support of B is a subspace of the support of A. If A is not null, then (3.2) is equivalent to the condition

$$AB = A \cdot B\,. \tag{3.3}$$

This relation is intrinsic to the definition of dual by (2.19), for every blade in $\mathcal{G}_n$ is a divisor of the pseudoscalar I.

We define the *join* J of blades A and B (to within a scalar factor) as a common dividend of lowest step. The support of J is then the usual 'lattice join' of the supports of A and B. We define J explicitly by

$$J = A \wedge B \tag{3.4}$$

when $A \wedge B$ does not vanish. The outer product of two blades is nonvanishing if and only if their supports have zero intersection. If the outer product of two blades is zero, then they have a nonscalar common factor and their greatest common divisor is well defined up to a scalar factor. When Step $A +$ Step $B \geq n$ and the join of A and B can be identified with the unit pseudoscalar I of $\mathcal{V}_n$, the scale of the join is fixed by a choice of scale for I. This is the case that occurs most frequently in applications to projective geometry.

For blades A and B with a common factor, we define the *meet* $A \vee B$ by the 'deMorgan' rule

$$(A \vee B)^{\widetilde{}} = \widetilde{A} \wedge \widetilde{B}\,, \tag{3.5}$$

where it is understood that the dual is taken with respect to the join J of A and B, that is, the duals are defined by (2.19) with $I = J$. Of course, this definition will not apply if J is a null blade, but that will not occur unless A or B is null, and we need not consider that possibility for the purpose of projective geometry.

The meet is a common factor of A and B with greatest step. Its support is the intersection of the supports of A and B. Thus, the meet and join provide the desired representation in geometric algebra of the basic operations in the algebra of subspaces of a vector space. For that purpose, by themselves the meet and join need be defined only up to a scale factor. But when a scale is chosen for the join, a scale for the meet is determined by (3.5).

Let r and s be the steps of blades A and B, respectively. If the join of A and B is the pseudoscalar I, then $r + s \geq n$. Unless stated otherwise, we henceforth take this condition for granted wherever the meet is expressed explicitly in terms of duality. The duality relations (2.19), (2.20) and (2.21) enables us to derive from (3.5) expressions for the meet which are more useful in computations. Thus, we obtain

$$A \vee B = (\widetilde{A} \wedge \widetilde{B})I = \widetilde{A} \cdot B = (-1)^{s(n-s)} A \cdot \widetilde{B}\,. \tag{3.6}$$

From this, we obtain for three blades

$$A \vee B \vee C = (\widetilde{A} \wedge \widetilde{B} \wedge \widetilde{C})I\,, \tag{3.7}$$

which shows that (under the above conditions on steps) the meet has the associative and anticommutative properties of the outer product.

The formula $A \vee B = \widetilde{A} \cdot B$ is the most generally useful expression for the meet because it makes the powerful expansion formula (2.16) available. However, in the important case when $r = s = n - 1$, both $\widetilde{A}$ and $\widetilde{B}$ are vectors, so $A \wedge B = A \times B$ and the meet can be expressed in terms of the commutator product by

$$A \vee B = (-1)^{n-1}(A \times B)I^{-1}\,, \tag{3.8}$$

Then properties of the commutator product can be exploited.

We could, of course, take $A \vee B = \widetilde{A} \cdot B$ as the definition of the meet. In fact, $\widetilde{A} \cdot B$ can be identified with Grassmann's *regressive product* (or, rather, Whitehead's generalization of it to arbitrary signature [20]). When $r + s = n$, $\widetilde{A} \cdot B$ is scalar valued, as noted in (2.22). For that case we define $A \vee B = 0$, since the meet must be a blade to represent a nonzero subspace. Accordingly, (3.5) and (3.6) define the meet only for $r + s > n$.

3.2. PROJECTIVE INTERPRETATION FOR BLADES

We adopt the standard identification of points in the projective space $\mathcal{P}_{n-1}$ with rays in $\mathcal{V}_n$. Each ray is the 1-dimensional solution set of the equation $x \wedge p = 0$ determined by a nonzero vector p, so we identify p as a representation of the unique ray it determines by referring to p as a *point*. Vectors p and q are regarded as *identical points* if and only if $p \wedge q = 0$, that is, $pq = p \cdot q$.

Each *projective line* in $\mathcal{P}_{n-1}$ is the 2-dimensional vector solution set of the equation $x \wedge A = 0$ determined by a 2-blade A. Accordingly, we can identify A with the unique line it determines. Blades A and B of step 2 are regarded as *identical lines* if and only if $AB = A \cdot B$, which is equivalent to the condition that they differ only by a scale factor. In the analogous way, we identify 3-blades with *projective planes* and $(n-1)$-blades (pseudovectors) with *hyperplanes*. Each pseudovector $N = \widetilde{N}I$ is the dual of a vector $\widetilde{N}$, so the duality relation (2.20) enables us to replace the equation $x \wedge N = 0$ for a hyperplane by the equivalent equation

$$x \cdot \widetilde{N} = 0\,. \tag{3.9}$$

Note that the inner product does not perform a metrical function here.

The theory of duality in $\mathcal{G}_n$ with respect to a nondegenerate inner product is the algebraic equivalent of the theory of *poles and polars* in $\mathcal{P}_{n-1}$ with respect to the surface of the second degree, or quadric, determined by this inner product. This sheds some light on the significance of (3.9). The inner product, indeed, has no metrical function here; it reflects how points and hyperplanes and their incidence relation are represented in geometric algebra in terms of duality, or, geometrically speaking, in terms of polarity. In particular, x and N are incident if and only if x and $\widetilde{N}$ are *conjugate* with respect to the quadric determined by the inner product. Furthermore, self-conjugate points are null and are exactly the points lying on this quadric, that is, the equation of this quadric is given by $x \cdot x = x^2 = 0$. This situation allows straightforward proofs of theorems involving polarity with respect to quadrics of any signature without referring to coordinate representations of quadratic forms (for examples, see Sections 4.3 and 5.6).

The relations among points, lines, planes, etc., in projective geometry can now be readily expressed by the various products in geometric algebra. For example, each pair of distinct points a and b determine a unique line $a \wedge b$. Also, a point p lies on a line A if and only if

$$p \wedge A = 0\,. \tag{3.10}$$

The same relation can be expressed in terms of the meet as the ‘absorption relation’

$$p \vee A = p\,. \tag{3.11}$$

If this is regarded as a relation in the projective line $\mathcal{P}_1$, then A can be identified with the pseudoscalar and (3.6) allows us to write

$$p \vee A = \widetilde{p} \cdot A = \widetilde{p}A = p\,.$$

This can be solved for the point

$$\widetilde{p} = pA^{-1} = p \cdot A^{-1}$$

dual to p on the line A. This is the simplest case in the projective theory of poles and polars, namely the theory of involutions in $\mathcal{P}_1$. In $\mathcal{P}_1$, the point $\widetilde{p}$ is *conjugate* to the point

p with respect to the conic determined by the nondegenerate inner product on $\mathcal{V}_2$, for the points satisfy the *orthogonality condition*

$$p \cdot \widetilde{p} = 0\,.$$

There are two distinct cases. The first case, $A^2 < 0$, occurs for Euclidean or anti-Euclidean signature only. The second case, $A^2 > 0$, occurs for the other signature. In this case the line A contains exactly two distinct 'self-dual' points; such points are null, for $\widetilde{p} = p$ implies $\widetilde{p} \cdot p = p^2 = 0$. Given any nonnull point p, there is a unique dual point $q = \widetilde{p}^{-1}$ such that

$$A = q \wedge p = qp\,.$$

This completes our discussion of $\mathcal{P}_1$. Of course, the same relations hold on any nonnull line in $\mathcal{P}_{n-1}$.

3.3. INCIDENCE RELATIONS AMONG LINES

Having seen how to characterize incidence relations between points and lines with geometric algebra, we turn now to incidence relations among lines. We seek formulations which are independent of the dimension of the projective space, in so far as that is possible. Two distinct lines A, B intersect in a point if and only if

$$A \wedge B = 0\,. \tag{3.12}$$

The point of intersection p is given by

$$p = A \vee B = (A \times B)J^{-1}\,, \tag{3.13}$$

where J is the join of A, B. If $A = a \wedge a'$ and $B = b \wedge b'$, then, by (2.22), we have

$$A \vee B = \widetilde{A} \cdot B = (\widetilde{A} \cdot b)b' - (\widetilde{A} \cdot b')b = [\,aa'b\,]\,b' - [\,aa'b'\,]\,b\,. \tag{3.14}$$

The condition that lines A, B are concurrent with line C is

$$p \wedge C = (A \vee B) \wedge C = 0\,. \tag{3.15}$$

In $\mathcal{P}_2$, $J = I$ and necessarily commutes with C; so, together with (2.31),

$$(A \vee B) \wedge C = [(A \times B)J^{-1}] \wedge C = [(A \times B)C]J^{-1} = A \wedge (B \vee C)\,. \tag{3.16}$$

Therefore, in $\mathcal{P}_2$ the condition (3.15) that three lines be concurrent is equivalent to the condition

$$\langle\, ABC \,\rangle = 0\,. \tag{3.17}$$

This is especially convenient for computations, because the join is not involved. In projective spaces of dimension 3 or greater, (3.17) is necessary but not a sufficient condition for concurrence. To see what else is needed, note that

$$\begin{aligned}[J^{-1}(A \times B)] \wedge C &= \langle\, J^{-1}(A \times B)C \,\rangle_3 \\ &= J^{-1}\langle\, ABC \,\rangle + J^{-1} \times [(A \times B) \times C)]\,.\end{aligned}$$

Therefore, for concurrence of lines in any dimension, (3.17) must be supplemented by

$$J \times [(A \times B) \times C] = 0\,. \tag{3.18}$$

Thus, it appears that there is no way to avoid introducing the join in some capacity. To understand how (3.18) can be satisfied, it is instructive to write $C = q \wedge p = q \times p$ and use the identity (2.29) to get

$$(A \times B) \times C = (A \times B)p \cdot q - [\,(A \times B) \cdot p\,] \wedge q + [\,(A \times B) \wedge p\,] \cdot q\,.$$

Now, if $p = A \vee B$, then (3.15) gives $(A \times B) \cdot p = 0$ and $(A \times B) \wedge p = p^2 J^{-1}$, whence all three terms in this expansion of $(A \times B) \times C$ commute with J to satisfy (3.18).

4. Two-Dimensional Projective Geometry

Now we are prepared for the main business of this article, to show how the theorems of projective geometry can be formulated and proved with geometric algebra. To demonstrate the scope and the flexibility of geometric algebra, we give examples concerning three major themes in projective geometry. First, it is well known that certain geometric configurations, such as those of Pappus and Desargues, are instrumental for proving many important theorems. We shall see that these configurations can be regarded as geometric representations of identities in geometric algebra. Second, the projective generation of higher-order loci from projective primitive forms (such as ranges of points, pencils of lines, etc.) is a method characteristic of projective geometry to define complicated figures in terms of simpler ones. For the plane, we pick the simplest example illustrating this method, namely the projective generation of a conic. This leads to a proof of Pascal's theorem. It is followed by a short discussion of polar and self-polar triangles in which the elegance and simplicity of formulating the theory of poles and polars in terms of algebraic duality is immediately evident. Third, explicit constructions have always been a characteristic feature of projective geometry. Our treatment of the plane cubic shows how such constructions can be modeled with geometric algebra, leading eventually to an explicit equation for the curve in terms of the geometric elements that uniquely determine it.

Our first task is to prove the theorems of Desargues and Pappus. Proofs using Grassmann's progressive and regressive products have been given by Forder [5], and the same proofs in terms of meet and join products are given in [1, 2, 16]. We develop the proofs rather differently, in a way which we believe gives insight into how the proofs work, especially the role of duality. We wish to exhibit how the proofs depend on relations with diverse applications to physics as well as mathematics. This can be seen already in algebraic relations among three points, to which we turn first.

4.1. ALGEBRA OF POINTS AND LINES

Every bivector in $\mathcal{G}_3$ is a 2-blade, thus the only relevant primary objects of $\mathcal{P}_2$ are vectors and 2-blades, that is, points and lines. The join of three noncollinear points a, b, c is a plane

$$J = a \wedge b \wedge c\,. \tag{4.1}$$

The points are the intersections of three coplanar lines

$$A = b \wedge c = \widetilde{A}J, \quad B = c \wedge a = \widetilde{B}J, \quad C = a \wedge b = \widetilde{C}J\,. \tag{4.2}$$

These lines are dual in the plane to three points

$$\widetilde{A} = AJ^{-1} = \frac{b \wedge c}{a \wedge b \wedge c}\,, \quad \widetilde{B} = BJ^{-1} = \frac{c \wedge a}{a \wedge b \wedge c}\,, \quad \widetilde{C} = CJ^{-1} = \frac{a \wedge b}{a \wedge b \wedge c}\,. \tag{4.3}$$

In the standard parlance of projective geometry, the points $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ are the poles of the polars A, B, C with respect to the conic determined by the nondegenerate inner product.

The intersection of the lines A, B, C at the points a, b, c is expressed by the equations

$$a = C \vee B = (\widetilde{C} \wedge \widetilde{B})J, \quad b = A \vee C = (\widetilde{A} \wedge \widetilde{C})J, \quad c = B \vee A = (\widetilde{B} \wedge \widetilde{A})J\,. \tag{4.4}$$

In addition, the following relations (holding for any signature) are easy to prove

$$a \cdot \widetilde{B} = a \cdot \widetilde{C} = b \cdot \widetilde{A} = b \cdot \widetilde{C} = c \cdot \widetilde{A} = c \cdot \widetilde{B} = 0\,, \tag{4.5a}$$

$$a \cdot \widetilde{A} = b \cdot \widetilde{B} = c \cdot \widetilde{C} = 1\,, \tag{4.5b}$$

$$\langle\, CBA \,\rangle = (a \wedge b \wedge c)^2 = J^2\,, \tag{4.6}$$

$$\widetilde{C} \wedge \widetilde{B} \wedge \widetilde{A} = J^{-1}\,. \tag{4.7}$$

Although our central concern here is with projective geometry, it is worth noting that, for the case of Euclidean signature, numerous applications of the foregoing relations among three points are given in [9] using the language of geometric algebra. Note, for example, that the relation of $\widetilde{A}$ to b and c in (4.2) is exactly that of the conventional vector cross-product. Consequently, the mathematical language used here articulates smoothly with standard vector algebra of Gibbs so widely used in physics [10]. In Appendix A of [9], the main theorems of spherical trigonometry are derived with geometric algebra. There the points a, b, c represent vertices of a spherical triangle while the lines A, B, C represent its sides.

4.2. THE THEOREMS OF DESARGUES AND PAPPUS AND RELATED THEOREMS

For Desargues' theorem we introduce another set of three points a', b', c', which determine a plane

$$J' = a' \wedge b' \wedge c' \tag{4.8}$$

and three lines

$$A' = b' \wedge c' = \widetilde{A}'J', \quad B' = c' \wedge a' = \widetilde{B}'J', \quad C' = a' \wedge b' = \widetilde{C}'J'\,. \tag{4.9}$$

It may be possible to prove Desargues' theorem from a single identity that covers all cases in $\mathcal{P}_3$. However, we consider only the case where J and J' represent the same projective plane $\mathcal{P}_2$. Then

$$J = a \wedge b \wedge c = [\,abc\,]I, \qquad J' = a' \wedge b' \wedge c' = [\,a'b'c'\,]I\,, \tag{4.10}$$

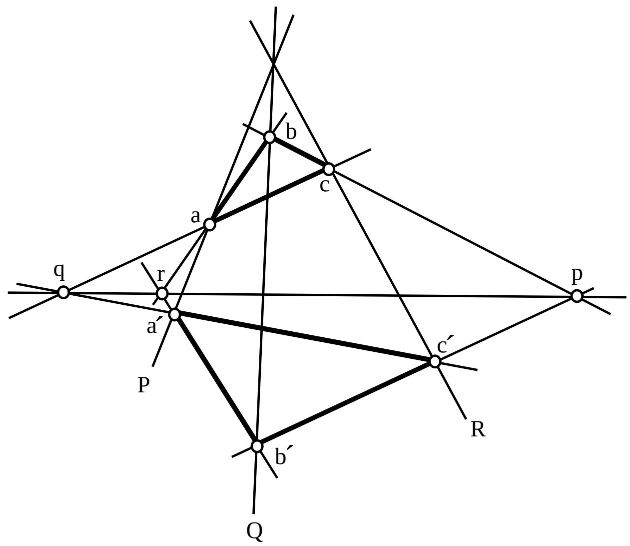

Fig.1. Desargues configuration.

where I is the unit pseudoscalar, so all points and lines commute with J and J'. This simplifies manipulations considerably.

The two sets of three points above determine three lines P, Q, R defined by

$$P = a \wedge a', \quad Q = b \wedge b', \quad R = c \wedge c' . \tag{4.11}$$

Similarly, the two sets of three lines intersect in three points p, q, r given by

$$\begin{aligned}(A \vee A')I &= A \times A' = (\widetilde{A} \wedge \widetilde{A}')JJ' = pI ,\\ (B \vee B')I &= B \times B' = (\widetilde{B} \wedge \widetilde{B}')JJ' = qI ,\\ (C \vee C')I &= C \times C' = (\widetilde{C} \wedge \widetilde{C}')JJ' = rI .\end{aligned} \tag{4.12}$$

Desargues' theorem says that the three points p, q, r are collinear if and only if the three lines P, Q, R are concurrent (Figure 1). The theorem follows immediately from the identity

$$\langle\, (a \wedge a')(b \wedge b')(c \wedge c')\,\rangle = JJ'\langle\, (\widetilde{A} \wedge \widetilde{A}')(\widetilde{B} \wedge \widetilde{B}')(\widetilde{C} \wedge \widetilde{C}')\,\rangle . \tag{4.13}$$

Note the symmetry with respect to duality. Using (4.11) and (4.12) the identity can be cast in the form

$$[\, pqr\,] = (JJ')^2 \langle\, PQR\,\rangle . \tag{4.14}$$

The right side of this identity vanishes if and only if the left side vanishes. This is an algebraic expression of Desargues' theorem.

The identity (4.13) can be derived from the fundamental identity (2.32) by using duality. Thus, (2.32) gives us

$$\langle\, a \wedge a' \;\, b \wedge b' \;\, c \wedge c'\,\rangle = (b' \wedge b \wedge c) \cdot (a \wedge c' \wedge a') - (b \wedge b' \wedge c') \cdot (c \wedge a \wedge a') .$$

By duality,

$$b' \wedge (b \wedge c) = (\widetilde{A}' \wedge \widetilde{C}'J') \wedge (\widetilde{A}J) = (\widetilde{A}' \wedge \widetilde{C}' \wedge \widetilde{A})JJ' ,$$

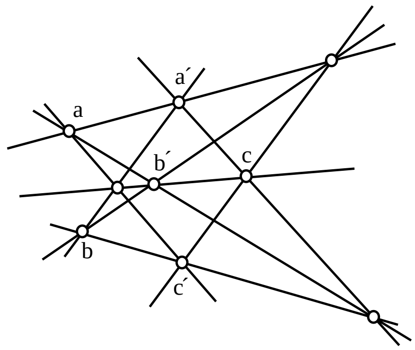

Fig. 2a Pappus configuration

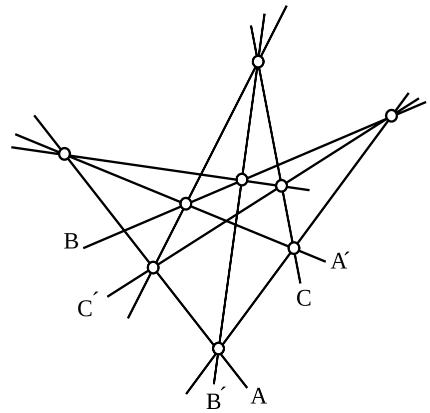

Fig. 2b Pappus configuration

and similarly,

$$a \wedge c' \wedge a' = (\widetilde{C} \wedge \widetilde{B} \wedge \widetilde{B}')JJ', \quad b \wedge b' \wedge c' = (\widetilde{A} \wedge \widetilde{C} \wedge \widetilde{A}')JJ' ,$$
$$c \wedge a \wedge a' = (\widetilde{B} \wedge \widetilde{C}' \wedge \widetilde{B}')JJ' .$$

Hence,

$$\langle\, a \wedge a' \;\; b \wedge b' \;\; c \wedge c' \,\rangle = -JJ'[(\widetilde{B} \wedge \widetilde{B}' \wedge \widetilde{C}') \cdot (\widetilde{C} \wedge \widetilde{A} \wedge \widetilde{A}') - (\widetilde{B}' \wedge \widetilde{B} \wedge \widetilde{C}) \cdot (\widetilde{C}' \wedge \widetilde{A}' \wedge \widetilde{A})] .$$

Application of (2.32) once more yields (4.13) as desired, so our proof of Desargues' theorem is complete. The theorem of Pappus says that if the three lines in any two of the sets of lines

$$\{a \wedge a,\, b \wedge b',\, c \wedge c'\}, \qquad \{a \wedge b',\, b \wedge c',\, c \wedge a'\}, \qquad \{a \wedge c',\, b \wedge a',\, c \wedge b'\}$$

concur, then the lines of the remaining set concur (Figure 2a). This follows immediately from the identity

$$\langle\, a \wedge a' \;\; b \wedge b' \;\; c \wedge c' \,\rangle + \langle\, a \wedge b' \;\; b \wedge c' \;\; c \wedge a' \,\rangle + \langle\, a \wedge c' \;\; b \wedge a' \;\; c \wedge b' \,\rangle = 0 , \tag{4.15}$$

which can be proved by applying the expansion (2.32) to each term. The dual relation for lines (Figure 2b) is given by the identity

$$\langle\, A \vee A' \;\; B \vee B' \;\; C \vee C' \,\rangle + \langle\, A \vee B' \;\; B \vee C' \;\; C \vee A' \,\rangle + \langle\, A \vee C' \;\; B \vee A'C \;\; \vee B' \,\rangle = 0 . \tag{4.16}$$

This follows directly from (4.15) by applying (4.13) and changing notation in accordance with (4.12).

Now we turn to an application of the algebraic form for Desargues' theorem. Given three points q, r, s on a line Z, a point p is called the *harmonic conjugate* of q with respect to r, s if there is a quadrangle $\{a, b, c, d\}$ (with no vertex incident with Z) such that pairs of opposite sides meet in r, s and the diagonals pass through p, q respectively (Figure 3). Desargues' theorem is instrumental for proving uniqueness of the harmonic conjugate. Suppose there is a quadrangle $\{a', b', c', d'\}$ not coinciding with $\{a, b, c, d\}$ such that the pairs of opposite sides pass through r, s and one diagonal, say $a' \wedge c'$, through q. We need to prove that the second diagonal $d' \wedge b'$ passes through p.

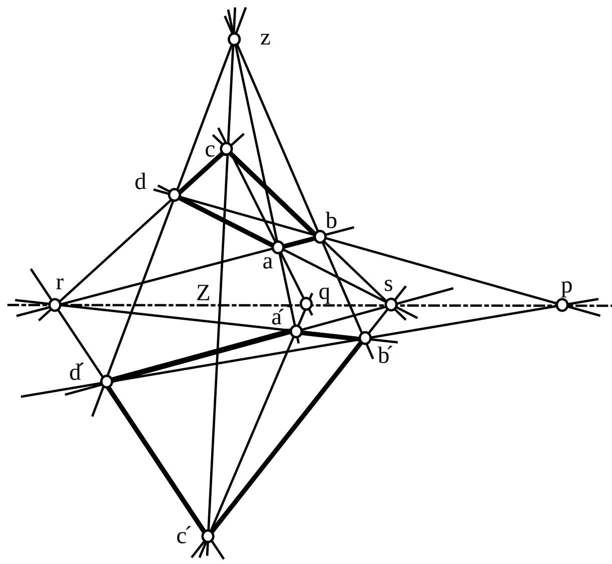

Fig. 3. Invariance of the construction of the fourth harmonic point.

In geometric terms, Desargues' theorem states that if two triangles are in perspective from a line, then they are perspective from a point and conversely. Suppose, the triangles $\{a, b, c\}$ and $\{a', b', c'\}$ as well as the triangles $\{a, d, c\}$ and $\{a', d', c'\}$ are in perspective from the same line (see Figure 3), then they are in perspective from the same point z and we conclude from (4.13) that

$$\langle\, a \wedge a'b \wedge b'c \wedge c'\,\rangle = \langle\, a \wedge a'd \wedge d'c \wedge c'\,\rangle = 0\,.$$

By (2.32) and (2.9), this is equivalent to the two relations

$$[\,aa'c\,][\,bb'c'\,] = [\,aa'c'\,][\,bb'c\,], \qquad [\,aa'c\,][\,dd'c'\,] = [\,aa'c'\,][\,dd'c\,]\,.$$

Taking the ratios of these two expressions and applying (2.32) again we obtain

$$\langle\, d \wedge d'b \wedge b'c \wedge c'\,\rangle = 0\,.$$

This says that the triangles $\{d, b, c\}$ and $\{d', b', c'\}$ are also in perspective from point z. They are therefore also in perspective from the line Z. It follows that $d' \wedge b'$ and $d \wedge b$ pass through the same point, so our proof is complete.

The next theorem we prove is due to Bricard [5, p. 63f.].

Consider two triangles $\{a, b, c\}$ and $\{a', b', c'\}$ with corresponding sides intersecting at points p, q, r. Then *Bricard's theorem* says that the meets of $a \wedge a'$, $b \wedge b'$, $c \wedge c'$ with the opposite sides $b \wedge c$, $c \wedge a$, $a \wedge b$ of triangle $\{a, b, c\}$ are collinear if and only if $p \wedge a'$, $q \wedge b'$, $r \wedge c'$ concur (Figure 4). We use the notation employed in our proof of Desargues' theorem. By (3.14) we have

$$P \vee A = [\,aa'b\,]c - [\,aa'c\,]b, \qquad Q \vee B = [\,bb'c\,]a - [\,bb'a\,]c,$$
$$R \vee C = [\,cc'a\,]b - [\,cc'b\,]a,$$

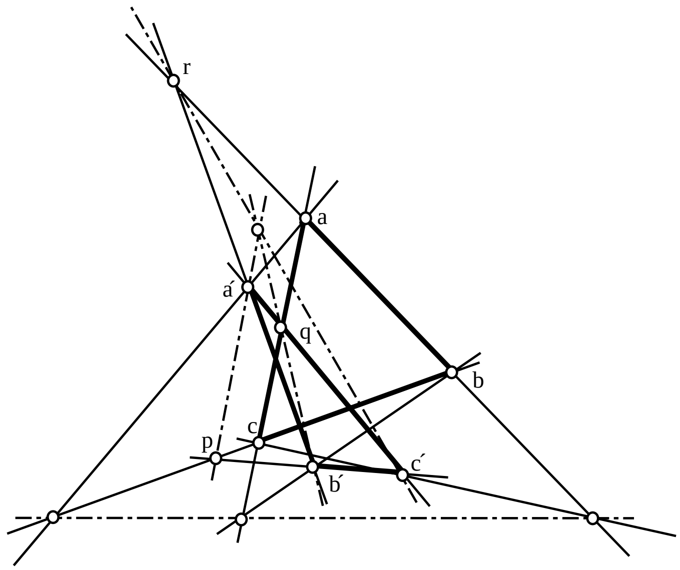

Fig. 4. Bricard's theorem.

hence

$$(P \vee A) \wedge (Q \vee B) \wedge (R \vee C) = J([aa'b][bb'c][cc'a] - [aa'c][bb'a][cc'b])\,.$$

In addition,

$$p \wedge a' = (A \vee A') \wedge a' = [bcb']c' \wedge a' - [bcc']b' \wedge a',$$
$$q \wedge b' = (B \vee B') \wedge b' = [cac']a' \wedge b' - [caa']c' \wedge b',$$
$$r \wedge c' = (C \vee C') \wedge c' = [aba']b' \wedge c' - [abb']a' \wedge c'\,,$$

therefore,

$$\langle\, p \wedge a' \;\; q \wedge b' \;\; r \wedge c' \,\rangle = \langle\, C'B'A' \,\rangle^2([aa'b][bb'c][cc'a] - [aa'c][bb'a][cc'b])\,,$$

and finally

$$(J')^2(P \vee A) \wedge (Q \vee B) \wedge (R \vee C) = J\langle\, p \wedge a' \;\; q \wedge b' \;\; r \wedge c' \,\rangle\,. \qquad (4.17)$$

This is the algebraic form of Bricard's theorem. In words, if the meets of the lines P, Q, R with the opposite sides of the triangle $\{a, b, c\}$ are collinear, then the joins of the points p, q, r with the opposite vertices of the triangle $\{a', b', c'\}$ concur.

4.3. CONICS

The set of all lines passing through a point is called a *pencil of lines*. Every such pencil is uniquely determined by two of its lines A, B and can be represented by the expression $A + \lambda B$ where $\lambda \in \Re \cup \{-\infty, +\infty\}$. The sum of lines is well-defined as bivector addition as long as the arbitrary scale factor is the same for all lines. Two pencils of lines $X = A + \lambda B$ and $X' = A' + \mu B'$ are said to be *projectively related* if they can be put in an ordered one-to-one correspondence such that X corresponds to X' if and only if $\lambda = \mu$. In this case,

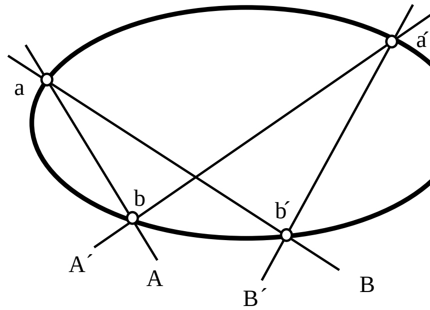

Fig. 5. Generation of a conic from two projective pencils of lines.

the set of intersection points of corresponding lines forms a conic. For, if $L = A + \lambda B$ and $L' = A' + \lambda B'$ intersect in x, we have

$$x \wedge L = x \wedge A + \lambda x \wedge B = 0\,,$$

$$x \wedge L' = x \wedge A' + \lambda x \wedge B' = 0\,,$$

and eliminating λ, we arrive at an equation of second order in x:

$$(x \wedge A)(x \wedge B') - (x \wedge B)(x \wedge A') = 0\,. \tag{4.18}$$

Alternatively, the conic is given by the parametric equation

$$\begin{aligned} x &= (A + \lambda B) \vee (A' + \lambda B') \\ &= A \vee A' + \lambda(A \vee B' + B \vee A') + \lambda^2 B \vee B'\,. \end{aligned}$$

Let $A = a \wedge b$, $B = a \wedge b'$, $A' = a' \wedge b$, $B' = a' \wedge b'$, so that $A \vee A' = b$, $B \vee B' = b'$ (Figure 5). Then letting $d = A \vee B' + B \vee A'$, the parametric equation takes the form

$$x = b + \lambda d + \lambda^2 b'\,.$$

Conversely, if $p \wedge q \wedge r \neq 0$, then the parametric equation $x = p + \lambda q + \lambda^2 r$ represents a nondegenerate conic. For, let $P = q \wedge r$, $Q = r \wedge p$, $R = p \wedge q$, then

$$p \wedge x = \lambda p \wedge q + \lambda^2 p \wedge r = \lambda(R - \lambda Q)$$

and

$$r \wedge x = r \wedge p + \lambda r \wedge q = Q - \lambda P$$

represent two projective pencils that generate the conic [5, §24].

From Equation (4.18), it follows that each one of the points a, a', b, b' lies on the conic of Figure 5, hence

$$[pab][pa'b'] = \mu[pab'][pa'b]$$

for some $\mu \neq 0$. If c' is some other point on the curve, then

$$\mu = \frac{[c'ab][c'a'b']}{[cab'][c'a'b]}\,,$$

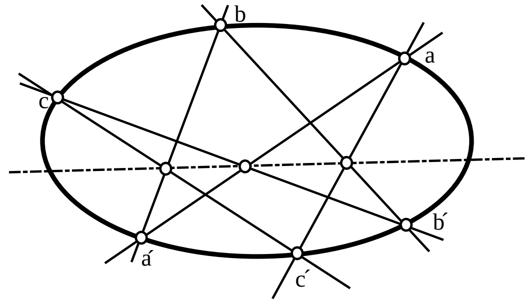

Fig. 6. Pascal's theorem.

and the equation for the conic takes the form

$$[pab][pa'b'][ab'c'][a'bc'] - [pab'][pa'b][abc'][a'b'c'] = 0\,. \tag{4.19}$$

This tells us that a unique conic is determined by five distinct points a, b, a', b', c'. Moreover, a point c lies on a conic passing through a, b, a', b', c' if and only if

$$[abc][ab'c'][a'bc'][a'b'c] - [a'b'c'][ab'c][a'bc][abc'] = 0\,. \tag{4.20}$$

It is readily verified that this is equivalent to the identity

$$((b' \wedge c) \vee (a' \wedge a)) \wedge ((a \wedge c') \vee (b' \wedge b)) \wedge ((a' \wedge b) \vee (c' \wedge c)) = 0\,. \tag{4.21}$$

This is an algebraic formulation of *Pascal's theorem* [5, p. 118], which says that the intersections of lines connecting opposite vertices of a hexagon lying on a conic are collinear (Figure 6).

Note that if $a \wedge b \wedge c = 0$ and $a' \wedge b' \wedge c' = 0$, the conic degenerates into two straight lines and Pappus's theorem emerges as a special case of Pascal's theorem.

Dualizing the above arguments leads to *Brianchon's theorem* and, again, to the dual of Pappus's theorem.

The polarity properties of a conic are automatically expressed by duality when we represent the conic by the (nondegenerate) inner product. Thus, a point x lies on the conic if and only if it is incident with its polar, $\widetilde{x} = xI^{-1}$, for

$$x \wedge \widetilde{x} = x \wedge (x \cdot I^{-1}) = (x \cdot x)I^{-1} = 0 \Leftrightarrow x^2 = 0\,,$$

in other words, if x is null. A point y is said to be *conjugate* to x if y is incident with $\widetilde{x}$. This is the symmetrical relation of *orthogonality*, for

$$y \wedge \widetilde{x} = x \wedge \widetilde{y} = 0 \Leftrightarrow x \cdot y = 0\,.$$

It is the foundation of the orthogonality concept within the context of projective metric geometry. Dually, the line equation for the conic is $X^2 = 0$ and two lines X, Y are conjugate if and only if $X \cdot Y = 0$.

We now prove a theorem which is instrumental in the synthetic treatment of the theory of poles and polars with respect to a conic. It says that if a quadrangle $\{a, b, c, d\}$ is inscribed

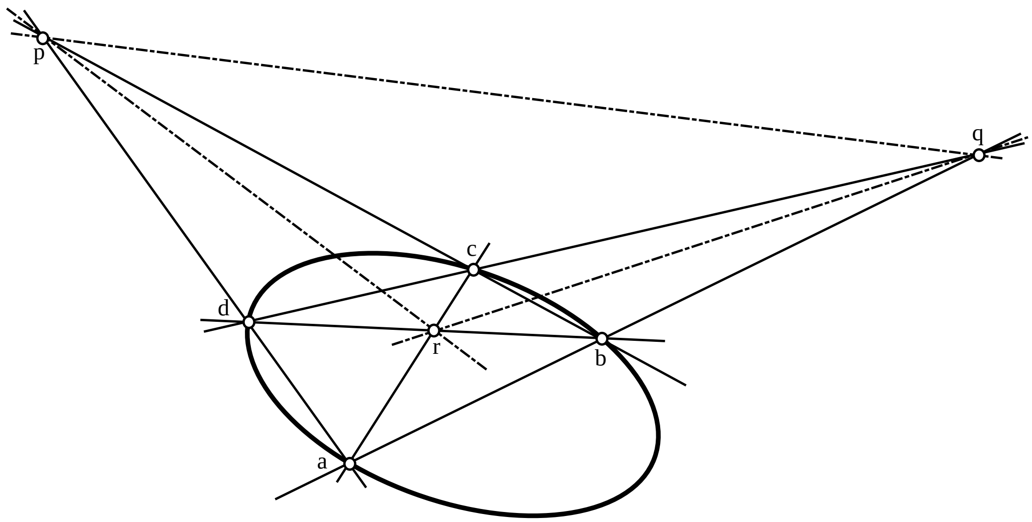

Fig. 7. Self-polar triangle of a quadrangle inscribed to a conic.

in a conic, then its diagonal triangle $\{p, q, r\}$ is self-polar (Figure 7). It is sufficient to show that the vertices of the diagonal triangle are pairwise conjugate. By (3.14), we have

$$p = (a \wedge d) \vee (c \wedge b) = [adc]b - [adb]c,$$

$$q = (a \wedge b) \vee (c \wedge d) = [abc]d - [abd]c,$$

$$r = (a \wedge c) \vee (b \wedge d) = [acb]d - [acd]b.$$

From this we conclude that

$$p \cdot q = q \cdot r = r \cdot p = -\tfrac{1}{2}(p^2 + q^2 + r^2)\,. \tag{4.22}$$

But the point $p + q + r = 2b[adc]$ lies on the conic, for

$$(p + q + r)^2 = p^2 + q^2 + r^2 + 2(p \cdot q + q \cdot r + r \cdot p) = 0\,.$$

This can only be true if $p \cdot q = q \cdot r = r \cdot p = 0$, that is, if $\{p, q, r\}$ is a self-polar triangle.

If a triangle $\{a, b, c\}$ is not self-polar, then it determines a unique polar triangle and they form a pair of Desargues triangles, which is to say that they are in perspective. Using the notation of Section 4.1, we conclude that $\{a, b, c\}$ and $\{\widetilde{A}, \widetilde{B}, \widetilde{C}\})$ (or $\{A, B, C\}$ and $\{\widetilde{a}, \widetilde{b}, \widetilde{c}\})$ are polar triangles (Figure 8). The intersection points p, q, r of corresponding sides are given by

$$p = \widetilde{a} \vee (b \wedge c) = (\widetilde{a} \wedge b)c - (\widetilde{a} \wedge c)b = (a \cdot bJ^{-1})c - (a \cdot cJ^{-1})b\,,$$

$$q = \widetilde{b} \vee (c \wedge a) = (\widetilde{b} \wedge c)a - (\widetilde{b} \wedge a)b = (b \cdot cJ^{-1})a - (b \cdot aJ^{-1})c\,,$$

$$r = \widetilde{c} \vee (a \wedge b) = (\widetilde{c} \wedge a)b - (\widetilde{c} \wedge b)a = (c \cdot aJ^{-1})b - (c \cdot bJ^{-1})a\,.$$

This implies that $p \wedge q \wedge r = [pqr]I = 0$, which proves the theorem that a pair of triangles which are not self-polar are in perspective.

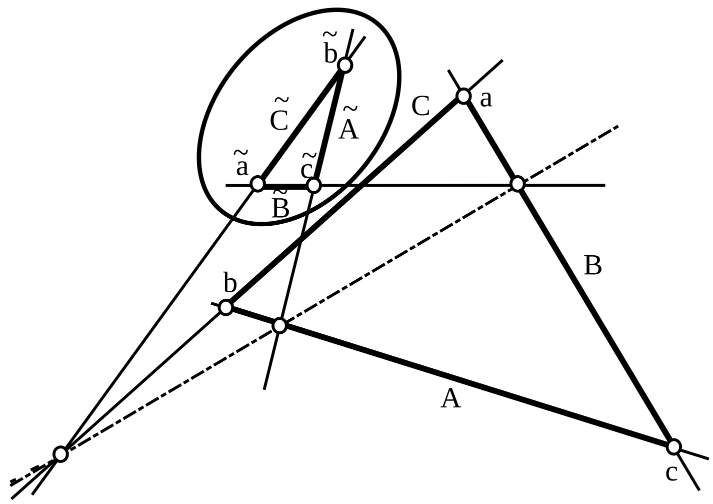

Fig. 8. Pair of polar triangles.

4.4. PLANE CUBICS

The following algebraic modeling of a synthetic algorithm for constructing points on a plane cubic curve was first outlined by Grassmann, and later refined and expanded by Whitehead [20] whom we follow here. This construction does not give a method to discover points on a plane cubic, but given one point x on the curve (determined by nine points in general position) the construction can be made.

Take A, B, C as three noncollinear lines and a, a_1, b, b_1, k, c as six points such that no three of them lie on a line and none of them is incident with A, B, or C (as, for example, in Figure 9). Consider any point x and the lines

$$R = ((x \wedge a) \vee A) \wedge a_1, \quad S = ((((x \wedge b) \vee B) \wedge k) \vee C) \wedge b_1, \quad T = x \wedge c\,;$$

then the locus of x is a cubic if R, S, T are concurrent, that is, if

$$\langle\, RST \,\rangle = 0\,. \tag{4.23}$$

This is necessarily the equation for a cubic since it is of the third degree in x. We need to prove that any plane cubic can be represented by an equation of this form. Since a plane cubic is uniquely determined by nine arbitrarily assumed points, it is sufficient to show that by a proper choice of the fixed lines and points of the construction mentioned above the cubic may be made to pass through any nine given points.

Before going into the construction itself, we analyze (4.23) in more detail. Let

$$p = R \vee T = (((x \wedge a) \vee A) \wedge a_1) \vee (x \wedge c) \quad \text{and} \quad q = (x \wedge b) \vee B\,. \tag{4.24}$$

Then (4.23) can be recast as the following relation between the points p and q

$$\begin{aligned} \langle\, RST \,\rangle = -\langle\, RTS \,\rangle &= -[p \wedge ((q \wedge k) \vee C) \wedge b_1] \\ &= [((q \wedge k) \vee C) \wedge b_1 \wedge p] = 0\,. \end{aligned} \tag{4.25}$$

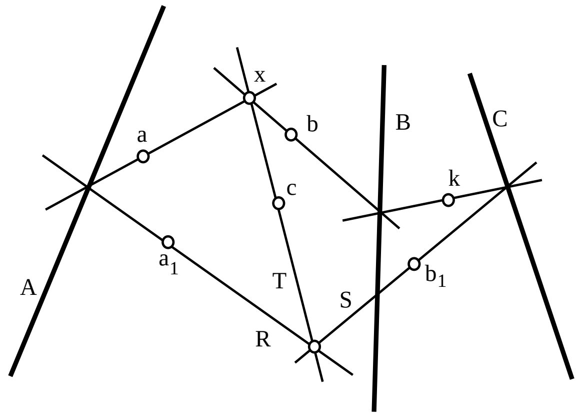

Fig. 9. Constructing algorithm for a plane cubic.

We note three properties of the curve.

(i) $p = 0$ if and only if x equals $a, c, (a_1 \wedge c) \vee A$, (or, of course, scalar multiples thereof).
This is obvious for $x = a$ or $x = c$. By (3.14) we have

$$p = [((x \wedge a) \vee A)a_1 x]c - [((x \wedge a) \vee A)a_1 c]x\,,$$

and

$$(x \wedge a) \vee A = [Aa]x - [Ax]a\,,$$

hence

$$p = -[Ax][aa_1 x]c - ([Aa][xa_1 c] - [Ax][aa_1 c])x\,.$$

Therefore, for $x \neq c$, $p = 0$ if and only if either $[Ax] = 0$ and $[xa_1 c] = 0$ or $[aa_1 x] = 0$ and $[Aa][xa_1 c] = [Ax][aa_1 c]$. In the first case, x lies on A and $a_1 \wedge c$, hence, we must have $x = (a_1 \wedge c) \vee A$. In the second case, $[aa_1 x] = 0$ implies that x lies on $a \wedge a_1$, so we can write $x = \lambda a + \mu a_1$. If we substitute this for x in the second condition, we get $\mu[Aa_1][aa_1 c] = 0$, hence $\mu = 0$ and $x = \lambda a$.

(ii) $p = x$ if and only if $[Ax] = 0$ or $[aa_1 x] = 0$; $q = x$ if and only if $[Bx] = 0$.
The solutions of $p = x$ are equivalent to the solutions of $p \wedge x = 0$ if $p \neq 0$, similarly for q. Thus, the above claims follow from the expressions

$$p \wedge x = -[Ax][aa_1 x]c \wedge x,$$

$$q \wedge x = ((x \wedge b) \vee B) \wedge x = ([Bh]x - [Bx]b) \wedge x = -[Bx]b \wedge x.$$

(iii) If A, B, C are concurrent, then their ppint of intersection, e, lies on the curve $\langle\, RST \,\rangle = 0$.
According to (ii), if $x = A \vee B$ or $x = (a \wedge a_1) \vee B$, then $p = x = q$. Hence, (4.25) can be cast in the form

$$\langle\, RST \,\rangle = ((x \wedge k) \vee C) \wedge b_1 \wedge x = -[Cx][kb_1 x]\,.$$

Therefore $\langle\, RST \,\rangle = 0$ implies either $[kb_1 x] = 0$ or $[Cx] = 0$. Hence, if the points $A \vee B$ and $(a \wedge a_1) \vee B$ lie on the cubic, then they must lie either on $k \wedge b_1$ or C. The last case implies (iii).

We are now ready to show that the plane cubic (4.23) can be made to pass through nine points $a, b, c, d, e, f, g, h, i$ no three of which lie on a line. Obviously, the cubic goes through a, b, c. Since $(a_1 \wedge C) \vee A$ lies on the cubic according to (i), we define $d = (a_1 \wedge c) \vee A$. Let A, B, C be concurrent in e, then (iii) implies that e is on the cubic. This allows us to write $A = d \wedge e$. From (iii), we conclude that if $(a \wedge a_1) \vee B$ lies on $b_1 \wedge k$ then it lies on the cubic. Accordingly, we define $f = (a \wedge a_1) \vee B$ such that $f \wedge b_1 \wedge k = 0$, and we may now write $B = e \wedge f$, since both e and f lie on B. Expanding $d = (a_1 \wedge c) \vee A$ and $f = (a_1 \wedge a) \vee B$ according to (3.14) implies $[a_1 cd] = 0$ and $[a_1 af] = 0$, respectively; hence a_1 lies both on $c \wedge d$ and $a \wedge f$ and we may write $a_1 = (a \wedge f) \vee (c \wedge d)$.

Now we substitute g, h, i respectively for x in the expressions (4.24) for p and q to get six points g_1, h_1, i_1 and g_2, h_2, i_2 respectively, all of which can be obtained from the nine given points by linear constructions. For example,

$$g_1 = (((g \wedge a) \vee A)a_1) \vee (g \wedge c) \qquad \text{and} \qquad g_2 = (g \wedge b) \vee B\,.$$

The points p and q are related by (4.25), hence we have

$$((g_2 \wedge k) \vee C) \wedge b_1 \wedge g_1 = 0, \tag{4.26a}$$

$$((h_2 \wedge k) \vee C) \wedge b_1 \wedge h_1 = 0, \tag{4.26b}$$

$$((i_2 \wedge k) \vee C) \wedge b_1 \wedge i_1 = 0. \tag{4.26c}$$

These relations determine k, C, b which were only partially determined by the conditions $[Ce] = 0$ and $[fb_1k] = 0$. Let C and k be such that they satisfy (4.26c) without conditioning b_1. With the help of (3.16), (4.26c) can be cast in the form

$$i_2 \wedge k \wedge (C \vee (b_1 \wedge i_1)) = i_2 \wedge k \wedge ([Cb_1]i_1 - [Ci_1]b_1) = 0\,.$$

In order not to specify b_1 we must have $[Ci_1] = 0$, that is, C passes through i_1, and since e lies also in C, we must assume that $C = e \wedge i_1$. It follows that k is determined by $[ki_1i_2] = 0$. But we showed already that $[fb_1k] = 0$, hence k is the intersection of $i_1 \wedge i_2$ and $f \wedge b_1$ where b_1 is as yet any arbitrarily assumed point.

Combining $[fb_1k] = 0$ with the relations (4.26a,b) we see that k must be such that the three lines $k \wedge f$, $((g_2 \wedge k) \vee C) \wedge g_1$, $((h_2 \wedge k) \vee C) \wedge h_1$ intersect in the same point b_1, hence we can write $b_1 = (((g_2 \wedge k) \vee C) \wedge g_1) \vee (k \wedge f)$. In addition, k lies on $i_1 \wedge i_2$ and therefore k is one of the points in which $i_1 \wedge i_2$ intersects the locus

$$\langle\, x \wedge f((x \wedge g_2) \vee C) \wedge g_1((x \wedge h_2) \vee C) \wedge h_1 \,\rangle = 0\,. \tag{4.27}$$

It can be proved that the cubic curve represented by (4.27) degenerates into three straight lines, namely B, C, and

$$K = \{(((h_1 \wedge g_1) \vee C) \wedge g_2) \vee (f \wedge h_1)\} \wedge \{(((g_1 \wedge h_1) \vee C) \wedge h_2) \vee (f \wedge g_1)\}$$

and that if k is made to lie on B or C, then the cubic (4.23) degenerates into a straight line and a conic (see [20, p. 236f.]). The only case left is the case in which k lies in K. This implies that $k = (i_1 \wedge i_2) \vee K$ since $[ki_1i_2] = 0$. Accordingly, with these assumptions, Equations (4.26a,b,c) are satisfied.

To sum up, we have proved that Equation (4.23) represents a nondegenerate cubic curve which passes through nine arbitrarily assumed points $a, b, c, d, e, f, g, h, i$ provided that

$$A = d \wedge e, \quad B = e \wedge f, \quad C = e \wedge i,$$

$$a_1 = (a \wedge f) \vee (c \wedge d), \quad k = (i_1 \wedge i_2) \vee K,$$

$$b_1 = (((g_2 \wedge k) \vee C) \wedge g_1) \vee (k \wedge f),$$

where $g_1, h_1, i_1, g_2, h_2, i_2$ and K are determined by the linear constructions mentioned above.

Note that the linear construction leading to (4.23) is satisfied by any point x on the cubic and represents a ten-cornered figure $x, a, b, c, d, e, f, g, h, i$ inscribed in the cubic. It is the analogue for plane cubics of Pascal's theorem for conics. The corresponding theorem for space cubics will be derived in Section 5.3.

5. Three-Dimensional Projective Geometry

The most prominent configurations in three-dimensional projective geometry involve tetrahedrons; two examples are given in Section 5.1. The projective method for generating higher-order loci from projective primitive forms in terms of geometric algebra is exhibited in a fairly extensive treatment of reguli, twisted cubics, and quadrics in Sections 5.3 through 5.5. In Section 5.6 we treat classical line geometry as simply the geometry of bivectors in the geometric algebra $\mathcal{G}_4$.

The use of algebraic identities to prove theorems of projective geometry owes a great deal to H. W. Turnbull (see, for example, [19]). Much of Turnbull's work was absorbed and reformulated in terms of Grassmann algebra by Forder [5]. The same has been done for the projective theory of reguli, twisted cubics, and quadrics by Whitehead [20] and Mehmke [14]. Forder provides a rich source of examples demonstrating the power and flexibility of Grassmann algebra. For this reason his book is particularly useful, although it does not go beyond Whitehead, or for that matter, Grassmann, in any important aspect of the theory proper.

5.1. ALGEBRA OF POINTS, LINES, AND PLANES

Points, lines, and planes in $\mathcal{P}_3$ are represented by vectors, 2-blades, and 3-blades respectively in $\mathcal{G}_4$. By duality, a sum of 3-blades in $\mathcal{G}_4$ can always be reduced to a 3-blade. However, a bivector generated by adding 2-blades in $\mathcal{G}_4$ cannot generally be reduced to a 2-blade. This is the fundamental fact underlying classical line geometry.

A point p or a line A lies on a plane Φ if and only if $p \vee \Phi = p$ or $A \vee \Phi = A$, respectively. The line of intersection of two planes $\Phi = a \wedge b \wedge c$ and $\Phi = a' \wedge b' \wedge c'$ is given by their meet $\Phi \vee \Phi'$ which can be expanded according to (2.16) and (2.22):

$$\begin{aligned}\Phi \vee \Phi' &= \widetilde{\Phi} \cdot \Phi' = \widetilde{\Phi} \cdot (a' \wedge b' \wedge c') \\ &= [\Phi a'] b' \wedge c' + [\Phi b'] c' \wedge a' + [\Phi c'] a' \wedge b' \,. \end{aligned} \tag{5.1}$$

Similarly, the intersection point of a plane $\Phi = a \wedge b \wedge c$ and a line $A = p \wedge q$ is given by

$$\Phi \vee A = \widetilde{\Phi} \cdot A = \widetilde{\Phi} \cdot (p \wedge q) = [\Phi p]q - [\Phi q]p \tag{5.2}$$

or, equivalently,

$$\Phi \vee A = \widetilde{A} \cdot \Phi = \widetilde{A} \cdot (a \wedge b \wedge c) = [Aab]c + [Aca]b + [Abc]a\,. \tag{5.3}$$

Hence, we have the identity

$$[abcp]q - [abcq]p = [pqab]c + [pqca]b + [pqbc]a\,. \tag{5.4}$$

5.2. PAIRS OF TETRAHEDRONS

The three-dimensional analogue of Desargues' theorem says that if the joins of corresponding vertices of two tetrahedra concur, then the lines of intersection of corresponding face planes are coplanar. To prove this (see [5, p. 108f.]), take a, b, c, d and a', b', c', d' as the vertices of the two tetrahedra. If p is the center of perspectivity, we can write

$$\alpha' a' = p + \alpha a, \quad \beta' b' = p + \beta b, \quad \gamma' c' = p + \gamma c, \quad \delta' d' = p + \delta d\,.$$

Hence

$$\begin{aligned}\alpha'\beta'\gamma'\, a' \wedge b' \wedge c' &= (p + \alpha a) \wedge (p + \beta b) \wedge (p + \gamma c)\\ &= p \wedge (\beta\gamma b \wedge c - \alpha\gamma a \wedge c + \alpha\beta a \wedge b) + \alpha\beta\gamma a \wedge b \wedge c\,.\end{aligned}$$

Therefore, if a point x lies on the line

$$L = \beta\gamma b \wedge c - \alpha\gamma a \wedge c + \alpha\beta a \wedge b$$

and on the plane $a \wedge b \wedge c$, then x lies on the plane $a' \wedge b' \wedge c'$. In other words, the line L is the line of intersection of $a \wedge b \wedge c$ and $a' \wedge b' \wedge c'$. The remaining lines of intersection of corresponding face planes are

$$\gamma\delta c \wedge d - \beta\delta b \wedge d + \beta\gamma b \wedge c, \qquad \delta\alpha d \wedge a - \gamma\alpha c \wedge a + \gamma\delta c \wedge d,$$

$$\alpha\beta a \wedge b - \delta\beta d \wedge b + \delta\alpha d \wedge a,$$

and they all lie in the plane

$$\beta\gamma\delta b \wedge c \wedge d - \alpha\gamma\delta a \wedge c \wedge d + \alpha\beta\gamma a \wedge b \wedge d - \alpha\beta\gamma a \wedge b \wedge c\,.$$

This concludes the proof.

The following identity was first derived by Forder [5, p. 170f.] who called it the *Möbius identity:*

$$(b \wedge c \wedge a') \vee (c \wedge a \wedge b') \vee (a \wedge b \wedge c') \vee (a' \wedge b' \wedge c') +$$

$$+ (b' \wedge c' \wedge a) \vee (c' \wedge a' \wedge b) \vee (a' \wedge b' \wedge c) \vee (a \wedge b \wedge c) = 0\,. \qquad (5.5)$$

The first term on the left side can be calculated by using (5.1) and (5.2) successively to get

$$(a \wedge b \wedge c') \vee (a' \wedge b' \wedge c') = [abc'a']b' \wedge c' + [abc'b']c' \wedge a'$$

and

$$(c \wedge a \wedge b') \vee (a \wedge b \wedge c') \vee (a' \wedge b' \wedge c')$$
$$= -[abc'a'][cab'c']b' + [abc'b'][cab'c']a' - [abc'b'][cab'a']c'\,.$$

Finally, since

$$(x \wedge y \wedge z) \vee w = [xyzw] \quad \text{for any } x, y, z, w,$$

we have

$$(b \wedge c \wedge a') \vee (c \wedge a \wedge b') \vee (a \wedge b \wedge c') \vee (a' \wedge b' \wedge c')$$
$$= -[abc'a'][cab'c'][bca'b'] + [a'b'ca][c'a'bc][b'c'ab]\,.$$

Exchanging primed and unprimed points, the left side of this equation becomes equal to the second term on the left side of (5.5), while the right side changes its sign. Thus, (5.5) is proved.

The Möbius identity may be interpreted as follows: If the planes $b \wedge c \wedge a'$, $c \wedge a \wedge h'$, $a \wedge b \wedge c'$ meet at a point d lying on the plane $a' \wedge b' \wedge c'$, then the intersection point, say d', of the planes $b' \wedge c' \wedge a$, $c' \wedge a' \wedge b$, $a' \wedge b' \wedge c$ lies in the plane $a \wedge b \wedge c$. Cyclical permutations of the points in this theorem leads to another theorem, namely: If for two tetrahedra $\{a, b, c, d\}$ and $\{a', b', c', d'\}$ the vertices a, b, c, d lie in the face-planes $b' \wedge c' \wedge d'$, $c' \wedge d' \wedge a'$, $d' \wedge a' \wedge b'$, $a' \wedge b' \wedge c'$, respectively, and a', b', c' lie in $b \wedge c \wedge d$, $c \wedge d \wedge a$, $d \wedge a \wedge b$, respectively, then d' lies in $a \wedge b \wedge c$. This is to say that each tetrahedron has its vertices on the face-planes of the other. In other words, the two tetrahedra are inscribed and circumscribed to each other. Such tetrahedra are called *Möbius tetrahedra*.

5.3. REGULI

The set of all lines joining corresponding points in two projective ranges of points on skew lines is called a *regulus*. Let $a, a'b, b'c, c'$ be any pairs of corresponding points such that $c = a + b$ and $c' = a' + b'$. Then $a + \lambda b$ and $a' + \lambda b'$ correspond to each other for every $\lambda \in \Re \cup \{-\infty, +\infty\}$, and their join is

$$a \wedge a' + \lambda(a \wedge b' + b \wedge a') + \lambda^2 b \wedge b'\,.$$

If we set $A = a \wedge a'$, $B = b \wedge b'$, $C = c \wedge c'$, then

$$C = (a + b) \wedge (a' + b') = A + B + (a \wedge b' + b \wedge a')\,,$$

and the join of corresponding points is given by the linear combination

$$(1 - \lambda)A + (\lambda^2 - \lambda)B + \lambda C\,.$$

Therefore, the lines of a regulus, called *generators*, are linear combinations of any three lines in the system. Every point p on a generator of a regulus is given by a linear combination of the form

$$p = a + \lambda b + \mu(a' + \lambda b') \,. \tag{5.6}$$

Any line L that cuts three generators of a regulus cuts all of them. For if L intersects A, B, C, that is, if $L \wedge A = L \wedge B = L \wedge C = 0$, then L intersects all linear combinations of A, B, C. Such a line L is called a *directrix* of the regulus. If L, M, N are directrices of a regulus, then any linear combination of them is also a directrix. Thus L, M, N generate a regulus, called the *associated regulus*, of the regulus generated by A, B, C. It follows that A, B, C are directrices of the regulus generated by L, M, N; hence each regulus is the associated regulus of the other.

To determine the locus of points lying on the lines of a regulus generated by A, B, C, consider a point x on C and its directrix L, that is, set $x = L \vee C$. The plane $((x \wedge A) \vee B) \wedge C$ contains C and L; hence

$$(((x \wedge A) \vee B) \wedge C) \wedge x = 0 \,. \tag{5.7}$$

This is the desired equation for the locus.

5.4. TWISTED CUBICS

The locus of points in the intersection of corresponding planes in three projective pencils of planes is called a *twisted cubic*. That this locus is in fact a curve of the third order can be established as follows. Let $X = A + \lambda B$, $X' = A' + \lambda B'$, $X'' = A'' + \lambda B''$ with $\lambda \in \Re \cup \{-\infty, +\infty\}$ be corresponding planes in three projective pencils of planes. Their point of intersection is given by

$$X \vee X' \vee X'' = a + \lambda b + \lambda^2 c + \lambda^3 d \,,$$

where

$$a = A \vee A' \vee A'',$$

$$b = B \vee A' \vee A'' + A \vee B' \vee A'' + A \vee A' \vee B\text{''},$$

$$c = A \vee B' \vee B'' + B \vee A' \vee B'' + B \vee B' \vee A'' d = B \vee B' \vee B''.$$

In general, the curve cuts a plane Φ in three points determined by the equation

$$a \wedge \Phi + \lambda b \wedge \Phi + \lambda^2 C \wedge \Phi + \lambda^3 d \wedge \Phi = 0 \,.$$

The points a and d are intersection points of corresponding planes; hence they belong to the curve. They are arbitrary points on the curve, since we can choose any set of three corresponding planes to represent them.

The projection of the cubic onto a plane Ψ from an arbitrary point on the curve, say d, is determined by the equation

$$(a \wedge d) \vee \Psi + \lambda(b \wedge d) \vee \Psi + \lambda^2 (c \wedge d) \vee \Psi = 0 \,.$$

Since the coefficients in this quadratic equation for λ represent points, the equation represents a conic. Therefore, the projection of a twisted cubic from any of its points forms a quadratic cone. Projecting a twisted cubic from two of its points, we see that it is the intersection of two quadratic cones with a common generator.

Consider seven distinct points p, a, b, c, a', b', c' on a twisted cubic. Since the projection of the curve from point p forms a quadratic cone, the lines

$$(p \wedge b' \wedge c) \vee (p \wedge b \wedge c'), \quad (p \wedge c' \wedge a) \vee (p \wedge c \wedge a'),$$
$$(p \wedge a' \wedge b) \vee (p \wedge a \wedge b')$$

are coplanar, by Pascal's theorem. Therefore, the points

$$l = (b' \wedge c) \vee (p \wedge b \wedge c'), \quad m = (c \wedge a') \vee (p \wedge c' \wedge a),$$
$$n = (a' \wedge b) \vee (p \wedge a \wedge b')$$

are coplanar with p, that is, $l \wedge m \wedge n \wedge p = 0$ or, equivalently,

$$((b' \wedge c) \vee (p \wedge b \wedge c')) \wedge ((c \wedge a') \vee (p \wedge c' \wedge a)) \wedge ((a' \wedge b) \vee (p \wedge a \wedge b')) \wedge p = 0\,. \tag{5.8}$$

With respect to the heptagon $\{p, c', b, a', c, b', a\}$ inscribed to the twisted cubic, (5.8) expresses the fact that the lines $b' \wedge c$, $c \wedge a'$, $a' \wedge b$ meet their opposite planes $p \wedge c' \wedge b$, $a \wedge p \wedge c'$, $b' \wedge a \wedge p$ in the points l, m, n in such a way that p is incident with the plane $l \wedge m \wedge n$. The same line of argument applied to the cones which project the twisted cubic from a, b, c, a', b', or c' leads to the following theorem concerning seven points on a twisted cubic which is the analogue of Pascal's theorem for six points on a conic: The edges of a simple heptagon inscribed to a twisted cubic intersect their opposite faces in the vertices of another simple heptagon which is both inscribed and circumscribed to the given heptagon.

5.5. QUADRICS

Let us call the set of all lines or planes passing through point a a bundle with center a. Now, take A, B, C as three noncoplanar lines meeting at d and Φ, Ψ, Ω as three noncollinear planes meeting at $a \neq d$. Then $A + \lambda B + \mu C$ with $\lambda, \mu \in \Re \cup \{-\infty, +\infty\}$ represents an arbitrary line through d, that is, a bundle of lines with center d. Similarly, $\Phi + \lambda\Psi + \mu\Omega$ represents an arbitrary plane passing through a, that is, a bundle of planes with center a. The two bundles are said to be *projectively related* (or correlative) if each line of the first corresponds to a plane of the second with the same values of λ and μ. The locus of the intersection points

$$p = (A + \lambda B + \mu C) \vee (\Phi + \lambda\Psi + \mu\Omega) \tag{5.9}$$

of corresponding elements is a surface of the second degree, that is, a *quadric*.

Without loss of generality, we may assume that A passes through a and that Ψ, Ω pass through d; hence A, $a \wedge d$, and $\Psi \vee \Omega$ all represent the same line (Figure 10). If C is replaced by $B + \epsilon C$ and Ω by $\Psi + \epsilon\Omega$, then ϵ can be chosen to make the coefficient of $\lambda\mu$ in (5.9) vanish; for, with this substitution, the coefficient has the form $2B \vee \Psi + \epsilon(B \vee \Omega + C \vee \Psi)$, and $B \vee \Psi$, $B \vee \Omega$, and $C \vee \Psi$ are points coinciding with d. Then (5.9) takes the form

$$p = a + \lambda b + \mu c + (\lambda^2\alpha + \mu^2\beta)d\,, \tag{5.10}$$

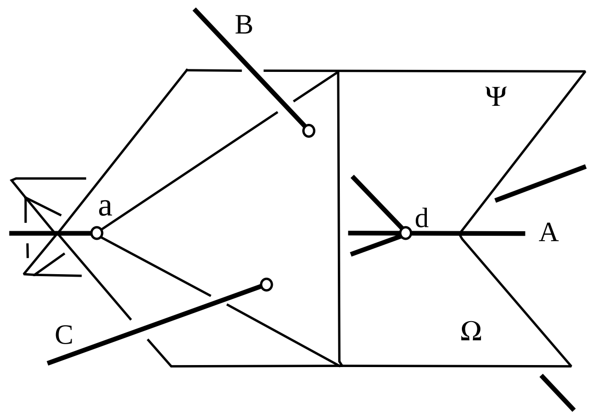

Fig. 10. Generation of a quadratic from two projective bundles.

where

$$a = A \vee \Phi, \quad b = B \vee \Phi, \quad c = C \vee \Phi, \quad \alpha d = B \vee \Psi, \quad \beta d = C \vee \Omega\,,$$

and α, β are fixed while λ, μ are variable. Of course, the points a and d lie on the surface given respectively by the values $\lambda = \mu = \infty$ and $\lambda = \mu = 0$ in (5.10).

It can be shown that this generation of a surface of the second degree does not depend on the choice of a and d (see [14, §63] or [5, §29.4]).

To prove that every surface with a parametric equation of the form (5.10) can be generated by two correlative bundles, we assume that $a \wedge b \wedge c \wedge d \neq 0$ and $\Phi = a \wedge b \wedge c$, $A = a \wedge d$, $B = b \wedge d$, $C = c \wedge d$. Let Ψ, Ω be two distinct planes through $a \wedge d$, so the free parameters are fixed such that $a = A \vee \Phi$, $b = B \vee \Phi$, $c = C \vee \Phi$, and $B \vee \Phi + C \vee \Psi = 0$. Then

$$p = (A + \lambda B + \mu C) \vee (\Phi + \lambda \Psi + \mu \Omega) = a + \lambda b + \mu c + \lambda^2 B \vee \Psi + \mu^2 C \vee \Omega\,,$$

where d, $B \vee \Psi$, and $C \vee \Omega$ represent the same point.

It is easy now to prove that the set of points lying on the generators of a regulus lie on a quadric. For, according to (5.6), any point on a generator of a regulus is of the form $p = a + \rho b + \sigma(c + \rho d)$. But this can be written as

$$p = a + \lambda(b + c) + \mu(b - c) + (\lambda^2 - \mu^2)d$$

where $\lambda = (\rho + \sigma)/2$ and $\mu = (\rho - \sigma)/2$.

Three linearly independent quadric loci have, in general, eight points in common and every quadric that passes through seven of these points is a linear combination of the given quadrics and, hence, contains the eighth point (see [17, p. 340]). These points are therefore symmetrically related and form a set of *eight associated points*. The problem of finding an explicit construction for the eighth associated point from seven arbitrary points was a rather famous challenge in the 19th century, and it has been solved in many different ways. Within geometric algebra, the problem is to find an algebraic criterion which determines whether or not eight given points form a set of associated points. A criterion of this kind was first determined by Turnbull [18].

According to (5.7), a quadric containing the lines L, M, N is determined by the equation

$$(((x \wedge L) \vee M) \wedge N) \wedge x = 0\,.$$

Let $M = a \wedge b$, then, by (5.2),

$$(x \wedge L) \vee (a \wedge b) = [xLa]b - [xLb]a\,,$$

so the equation for the quadric can be cast in the form

$$[Lax][Nbx] = [Lbx][Nax]\,. \tag{5.11}$$

When the condition that the entire line $a \wedge b$ lies in the quadric is relaxed, Equation (5.11) generalizes to

$$[Lax][Nbx] = \lambda[Lbx][Nax]\,. \tag{5.12}$$

Assuming that this quadric passes through a point c, λ is given by

$$\lambda = \frac{[Lac][Nbc]}{[Lbc][Nac]}\,,$$

and (5.12) becomes

$$[Lbc][Lax][Nac][Nbx] - [Lac][Lbx][Nbc][Nax] = 0\,. \tag{5.13}$$

This is the equation for a quadric passing through points a, b, c and lines L, N. Writing $L = a' \wedge b'$, $N = c' \wedge d'$ and setting $x = d$, we have

$$[a'b'bc][a'b'ad][c'd'ac][c'd'bd] - [a'b'ac][a'b'bd][c'd'bc][c'd'ad] = 0\,. \tag{5.14}$$

This is the desired relation which must be satisfied if the eight points $a, b, c, d, a', b', c', d'$ are to form a set of associated points.

5.6. LINE GEOMETRY

Classical line geometry is concerned with families of lines, especially in $\mathcal{P}_3$. To show how the subject can be developed in the language of geometric algebra, we discuss some corresponding families of 2-blades in $\mathcal{G}_4$. We limit our attention to the case of real scalars.

The development of line geometry in the 19th century was greatly influenced by the fact that the action of a single force on a rigid body can be represented geometrically by a unique oriented line (along which the force acts) and a positive scalar (for the magnitude of the force). Each such force (action) can be represented by a unique 2-blade in $\mathcal{G}_4$. Note that this differs from the representation of a projective line by a 2-blade only by fixing the scale of the blade. Thus, there is a one parameter family of forces associated with each line. The superposition of forces acting on a rigid body determines a unique resultant force, and this is expressed quantitatively by the addition of the corresponding 2-blades. This idea is abstracted and generalized in line geometry as a 'geometrical rule' for combining lines. It requires, however, the introduction of new geometric concepts, for the bivector sum of 2-blades in $\mathcal{G}_4$ is not necessarily reducible to a 2-blade. One geometrical name for such a bivector is *'screw,'* since it can be interpreted kinematically as a infinitesimal screw motion along a line (sometimes called a *twist*). A line can be regarded as a *degenerate screw*, represented by a bivector with only a single blade. The mechanical analogue of a screw is sometimes called a *wrench.* A wrench is a bivector in $\mathcal{G}_4$ representing the action of a system of forces on a rigid body. It represents a single resultant force only if it degenerates to a 2-blade. In general, a wrench represents a resultant force and couple (see Section 6.1 of [9]).

Generalized line geometry has been treated extensively in terms of Grassmann algebra by Whitehead [20]. He called it the 'non-metrical theory of forces.' His use of the obsolete term 'force' is unfortunate, because it distracts from the purely mathematical character of his treatment. His 'forces' are just general bivectors in $\mathcal{G}_4$, and his theory is concerned with systems of bivectors. It is 'nonmetrical' in its study of bivector properties which do not depend on signature. Our subject here is the same. We hope that our discussion will renew interest in Whitehead's important work, which has not received the attention it deserves. The ideas have applications far beyond classical mechanics, though the applications to mechanics are still of interest.

Any sum of coplanar lines is a line in the same plane. This relation is well-defined as bivector addition as long as the arbitrary scale factor is the same for all lines. In general, a linear combination of lines which are not coplanar is a screw (or bivector) which cannot be reduced to a line. However, every screw can be expressed as a sum of two lines. In fact, any screw S can be resolved into the sum of a line passing through a given point p and a line Iying in a given plane Π which does not contain p. This can be proved from the identity

$$(S \wedge p) \cdot \widetilde{\Pi} = S(p \cdot \widetilde{\Pi}) - (S \cdot \widetilde{\Pi}) \wedge p\,,$$

an application of (2.16). Adjusting the magnitude of the vector $\widetilde{\Pi} = \Pi I^{-1}$ so that $p \wedge \Pi = (p \cdot \widetilde{\Pi}) I = I$, we have

$$S = (S \cdot \widetilde{\Pi}) \wedge p + (S \wedge p) \cdot \widetilde{\Pi}\,. \tag{5.15}$$

The first term on the right is obviously a 2-blade because $S \cdot \widetilde{\Pi}$ is a vector, and the second term is a 2-blade because the trivector $S \wedge p$ is necessarily a blade in $\mathcal{G}_4$ even if S is not. Thus (5.15) is the desired resolution of an arbitrary screw S into two lines. The result can be expressed in terms of meet and join by writing $a = s \cdot \widetilde{\Pi} = -\widetilde{S} \cdot \Pi = -S \vee \Pi$ and $\Phi = -S \wedge p$ so that $(S \wedge p) \cdot \widetilde{\Pi} = -\Phi \cdot \widetilde{\Pi} = \widetilde{\Phi} \cdot \Pi = \Phi \wedge \Pi$. Then (5.15) takes the form

$$S = a \wedge p + \Phi \vee \Pi\,. \tag{5.16}$$

The screw S is a line if and only if the lines on the right side of this decomposition intersect.

A bivector S is a blade (or line) if and only if $S \wedge S = 0$. For if S is expressed as a linear combination

$$S = \lambda L + M \tag{5.17}$$

of lines L and M, then $S \wedge S = 2\lambda L \wedge M$, which vanishes if and only if L and M intersect. If $S \wedge S \neq 0$, then the lines L and M are said to be *conjugate* with respect to S.

Every line L has a unique conjugate M with respect to a given screw S. For if M is defined in terms of S and L by (5.17), then we require

$$M \wedge M = (S - \lambda L) \wedge (S - \lambda L) = 0\,,$$

which is satisfied only if λ has the unique value

$$\lambda = \frac{1}{2}\,\frac{S \wedge S}{L \wedge S}\,. \tag{5.18}$$

A line N is said to be a *null line* of a bivector S if $S \wedge N = 0$. (This is not to be confused with the more common metrical definition of a 'null line' used in relativity.) The set of all null lines of S is called a *linear complex* if $S \wedge S \neq 0$ or a *special linear complex* if $S \wedge S = 0$. In the latter case, S is a line and its null lines are those lines which intersect it. In the former case, if L is any line intersected by N, then the conjugate of L with respect to S is also intersected by N. This follows immediately from (5.17); thus, $M \wedge N = S \wedge N - \lambda L \wedge N = 0$. It follows that any line intersecting a pair of lines which are conjugate with respect to S is a null line of S. On the other hand, it is easily proved that if every null line which intersects the line L also intersects a line M, then L and M are conjugate.

Two screws S, S' are said to be *reciprocal* if $S \wedge S' = 0$. In this case, if L is a null line of S', then its conjugate M with respect to S is also a null line of S'. For by (5.17), $S' \wedge M = S' \wedge (S - \lambda L) = 0$.

The set of all linear combinations of two independent bivectors $S' \neq S$ is called a *pencil of bivectors*, or, in geometric parlance, a *pencil of linear complexes.* A pencil of bivectors contains two, one, or no lines according as

$$(S + \lambda S') \wedge (S + \lambda S') = S \wedge S + \lambda(S \wedge S' + S' \wedge S) + \lambda^2 S' \wedge S' = 0 \tag{5.19}$$

has two, one, or no real roots for λ. With respect to the pencil of linear complexes, these two uniquely determined lines, if they exist, are called *directrices* and represent the special linear complexes of the pencil.

A line which is a null line for every linear complex of a pencil is a null line for each of its special linear complexes; hence it intersects both directrices (if they exist). Conversely, every line which intersects both directrices is a common null line of all linear complexes of the pencil. From this we conclude that the directrices of a pencil of linear complexes are conjugate lines with respect to every complex of the pencil.

The concept of duality enables us to relate properties of linear complexes to properties of quadrics. If S is a bivector distinct from its dual $\widetilde{S} = SI^{-1}$ then $\widetilde{S}$ and S define a pencil of linear complexes with directrices determined by the equation

$$\begin{aligned}(S + \lambda\widetilde{S}) \wedge (S + \lambda\widetilde{S}) &= S \wedge S + 2\lambda(S \cdot \widetilde{S}) + \lambda^2 \widetilde{S} \wedge \widetilde{S} \\ &= S \wedge S + 2\lambda I(S \cdot S) + \lambda^2 I^2 S \wedge S \\ &= S \wedge S\left(1 + 2\lambda I\,\frac{S \cdot S}{S \wedge S} + \lambda^2 I^2\right) = 0\,.\end{aligned} \tag{5.20}$$

The unit pseudoscalar satisfies $I^2 = \pm 1$, depending on the signature. Therefore, the solutions of (5.20) are of the form $\pm\lambda_0$, $+1/\lambda_0$, and the two directrices D_1, D_2 are

$$D_l = S + \lambda_0 \widetilde{S}, \qquad D_2 = \pm\lambda_0 S + \widetilde{S}\,.$$

It follows that $\widetilde{D}_1 = D_2$ and $\widetilde{D}_2 = \pm D_1$; in other words, D_1 and D_2 are polar lines with respect to the quadric determined by the nondegenerate inner product. D_1 and D_2 exist and are distinct if and only if $(S \cdot S)^2 > I^2(S \wedge S)^2$; they are coincident if and only if $(S \cdot S)^2 = I^2(S \wedge S)^2$.

We are now able to prove a theorem which plays a significant role in the kinematics of projective metric spaces. (In fact, for Euclidean and hyperbolic spaces it implies the existence and uniqueness of a screw axis for any given moment of a motion.) The theorem says that in any linear complex S with $(S \cdot S)^2 > I^2(S \wedge S)^2$ there are exactly two lines which are conjugate with respect to S as well as polar with respect to a given quadric (determined by the inner product). In fact, these two lines are the directrices of the pencil determined by S and $\widetilde{S}$ (see above). If there were another such pairs of lines, say E and $\widetilde{E}$, which are not directrices of the pencil, then we would have $S = E + \mu\widetilde{E}$, hence $\widetilde{S} = \widetilde{E} \pm \mu E$. Accordingly, E and $\widetilde{E}$ would be directrices of the pencil, in contradiction to the assumption.

6. Discussion

Projective geometry can be developed using synthetic or analytic methods. The two methods seem so different that synthetic and analytic geometries have developed in parallel into what some regard as independent branches of mathematics. The analytic method turned out to be the more powerful of the two. While synthetic geometry has stagnated in the twentieth century, the analytic method has flowered through linear algebra to diverse applications throughout mathematics. Synthetic geometry has survived, nevertheless, because it has decided advantages. Unfortunately, those advantages are not readily available to mathematicians schooled only in the analytic method. We think that the gap between synthetic and analytic approaches should be regarded as a deficiency in the design of mathematical systems. The greater range and flexibility of the analytic approach suggests that this deficiency can be corrected by redesigning analytic tools and methods to accommodate synthetic ideas. We see our present formulations of projective geometry in terms of geometric algebra as a step in a redesign process started by Grassmann.

The coordinate-free formulation of projective geometry in terms of geometric algebra is clearly much closer to the synthetic formulation than the traditional formulation in terms of coordinates. It provides direct algebraic representations of the synthetic primitives 'point, line, plane' and their relations such as 'incident' and 'dual.' The theory of duality is one of the most powerful and yet elementary means to prove a host of theorems in projective geometry, so its algebraic counterpart should be equally powerful and elementary. The duality operator in geometric algebra satisfies this requirement perfectly, which cannot be said of the indirect representation for duality in conventional linear algebra.

Synthetic geometry is much more than a system of primitive elements and relations. It is a system of constructs with interpretations as geometrical structures which we can apprehend as conceptual units transcending their primitive constituents. The ultimate goal of the theory is to develop a hierarchy of geometrical structures based on a morphology of geometrical objects. Hence, an ideal algebraic system for modeling these characteristics should enable us to deal with geometrical objects directly without distracting reference to subordinate structures such as coordinate systems. Geometric algebra has this advantage over traditional approaches.

Synthetic geometry is intimately related to diagrammatic representations of geometric concepts. Only the simplest projective constructs, such as the configurations of Desargues and Pappus, can be fully represented by diagrams. However, it may be possible and it would certainly be desirable to develop new diagrammatic representations for more complex concepts. Of course, diagrams have no place in the formal theory, but it must be admitted that synthetic constructs and the diagrams that go with them are engines of discovery and development in the theory. How would one be led, for instance, to the rigorous algebraic formulation of Desargues' theorem without a diagram depicting the system of relations involved? We believe that in the pursuit of rigor the significance of diagrams in mathematics has been underestimated. Diagrams are as important in understanding and applying a mathematical theory as in developing it. We submit that synthetic constructs and their associated diagrams perform an essential organizing function within projective geometry. We propose this as a design principle for integrating the synthetic and analytic approaches.

The algebraic identities underlying the theorems of projective geometry are, of course, significant in metrical geometry as well. As ably exposed in [1], they belong to the vast system of identities in *classical invariant theory.* Since the formalism developed in [1] is subsumed by the formalism in this article, invariant theory can be given an efficient, coordinate-free formulation in terms of geometric algebra. To the end of unifying mathematics, there is a great need for organizing the multifarious results and applications of invariant theory, even after deciding that geometric algebra is the appropriate language. We see this as a need for elucidating the 'synthetic structure' of invariant theory, a generalization of the problem of integrating synthetic and algebraic approaches to projective geometry. To be more specific about what needs to be done, we note that miscellaneous relations between projective and metrical geometries have been noticed by many different people. For example, Fano and Racah [4] have observed that the graphical representation of angularmomentum recoupling coefficients in quantum mechanics has a 'Desarguesian structure' if triads of quantum numbers are represented by three points on a line. This has been further elaborated by Robinson [15] and Judd [13]. However, as Judd noted in his concluding remarks, 'the deeper issue of why projective geometry should play a role at all has not been discussed.' Indeed, such isolated fragments of the synthetic structure of invariant theory deserve to be studied systematically.

Appendix

The most comprehensive and systematic applications of Grassmann algebra to geometry are given by Whitehead [20] and Forder [5]. By and large they employ the notations of Grassmann. To facilitate the translations of these rich stores of theorems and examples into the language of geometric algebra, we provide the following brief guide with the nomenclature and notation of Whitehead in quotation marks.

An 'extensive magnitude of order r' is an r-vector; it is said to be 'simple' if it is a blade or 'compound' otherwise. The outer product of blades A and B is called the 'combinatorial product' and written '(AB)'. The 'supplement $|A$' of blade A is essentially its dual. In geometric algebra it can be defined by

$$|A = A^{\dagger} I \,, \tag{A1}$$

where I is the unit pseudoscalar. This is identical to Grassmann's supplement if a Euclidean signature is adopted so that $I^\dagger I = 1$. By (2.19),

$$| \, |A = (A^\dagger I)^\dagger I = I^\dagger A I = (-1)^{r(n-r)} A \,. \tag{A2}$$

For arbitrary signature the definition (A1) is identical to Whitehead's 'extended supplement.' Whitehead applied it to the theory of poles and polars with respect to quadrics determined by the corresponding inner product (see [20,§§108–124]).

Following Grassmann, Whitehead eventually uses the notation 'AB' for both 'regressive' and 'progressive' products, depending on context to remove the ambiguity. The 'regressive' product 'AB' of two blades A, B of step r, s with $r + s > n$ is defined in terms of the 'progressive product' of $|A$ with $|B$ by

$$\text{`}|AB = (|A \, |B)' \,. \tag{A3}$$

If $r + s < n$, then 'AB' denotes a 'progressive product' while '$|A \, |B$' is 'regressive,' as expressed by

$$\text{`}|(AB) = |A \, |B' \,. \tag{A4}$$

For $r + s = n$ both interpretations are allowed.

In our notation (A3) and (A4) correspond to the following relations

$$\begin{aligned} (A \vee B)^\dagger I &= (A^\dagger I) \wedge (B^\dagger I) \qquad \text{if } r + s > n \,, \\ (A \wedge B)^\dagger I &= (A^\dagger I) \vee (B^\dagger I) \qquad \text{if } r + s < n \,, \\ (A \wedge B)^\dagger I &= (A \vee B)^\dagger I = [AB] \quad \text{if } r + s = n \,. \end{aligned} \tag{A5}$$

Whitehead expands 'regressive products' using his 'extended rule of the middle factor' [20, §103] which is equivalent to using (2.16) to expand (2.17) with $B = I$.

Multiple products in Grassmann's notation must be read from left to right. The rules given above uniquely determine whether a given product between two blades is 'progressive' or 'regressive.' Grassmann avoids parentheses by introducing periods as markers. As examples showing how, take a, b, c, d, x as points and A, B, C, D, X as lines in a plane, and consider the translations

$$x = \text{`}ABa \cdot bc' = ((A \vee B) \wedge a) \vee (b \wedge c),$$

$$X = \text{``}ABaCbDc\text{''} = (((((A \vee B) \wedge a) \vee C) \wedge b) \vee D) \wedge c,$$

$$\begin{aligned} &\text{`}(xaAB)(xcAb \cdot ad \cdot xc)\text{'} \\ &\quad = \{((x \wedge a) \vee A) \wedge B\} \, \{((((x \wedge c) \vee A) \wedge b) \vee (a \wedge d)) \wedge (x \wedge c)\} \\ &\quad = [((x \wedge a) \vee A)B] \, \langle \, ((x \wedge c) \vee A) \wedge b \;\, a \wedge d \;\, x \wedge c \rangle. \end{aligned}$$

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