Old Wine in New Bottles: A new algebraic framework for computational geometry

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1 Introduction

My purpose in this chapter is to introduce you to a powerful new algebraic model for Euclidean space with all sorts of applications to computer-aided geometry, robotics, computer vision and the like. A detailed description and analysis of the model is soon to be published elsewhere [1], so I can concentrate on highlights here, although with a slightly different formulation that I find more convenient for applications. Also, I can assume that this audience is familiar with Geometric Algebra, so we can proceed rapidly without belaboring the basics.

2 Minkowski Algebra

Let $\mathcal{R}_{p,q} = \mathcal{G}(\mathcal{R}^{p,q})$ denote the Geometric Algebra generated by a vector space $\mathcal{R}^{p,q}$ with non-degenerate signature (p, q), where p is the dimension of its largest subspace of vectors with positive signature. The signature is said to be *Euclidean* if $q = 0$ and *Minkowski* if $q = 1$. We will be concerned with the *Minkowski algebra* $\mathcal{R}_{n+1,1}$ and its Euclidean subalgebra $\mathcal{R}_n$ determined by a *designated* unit bivector (or blade) E.

First we consider the Minkowski plane $\mathcal{R}^{1,1}$ and the Minkowski algebra $\mathcal{R}_{1,1} = \mathcal{G}(\mathcal{R}^{1,1})$ that it generates. It is most convenient to introduce a null basis $\{e, e_*\}$ for the plane that satisfies

$$e^2 = e_*^2 = 0, \quad e \cdot e_* = 1. \tag{1}$$

This generates a basis $\{1, e, e_*, E\}$ for $\mathcal{R}_{1,1}$, where

$$E = e \wedge e_* \quad \implies \quad E^2 = 1 \tag{2}$$

defines a unit pseudoscalar for the plane. It is of some interest to remark that the "∗" notation has been adopted to indicate that e_* is "dual" to e in the sense of linear forms. That simplifies comparison of alternative mathematical representations.

Using the expansion $ee_* = e \cdot e_* + e \wedge e_*$, we can express the basis relations in the convenient form

$$ee_* = 1 + E, \tag{3}$$

whence

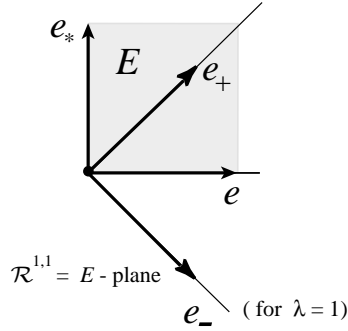$$e_*e = 1 - E = 1 + E^\dagger = (ee_*)^\dagger. \tag{4}$$

Fig. 1

We can also derive the "absorption property" for null vectors:

$$Ee = -eE = e, \qquad e_*E = -Ee_* = e_*. \tag{5}$$

The above relations suffice for all our dealings with the E-plane. However, it is of some interest to compare the null basis with an orthonormal basis defined by

$$e_{\pm} \equiv \frac{1}{\sqrt{2}}(\lambda e \pm \lambda^{-1}e_*), \quad \lambda \neq 0. \tag{6}$$

We note that

$$e_{\pm}^2 = \pm 1, \qquad e_+ \cdot e_- = 0, \qquad E = e \wedge e_* = e_+ \wedge e_- = e_+e_- \tag{7}$$

The two sets of base vectors are in Fig. 1 for $\lambda = 1$. As λ varies in eqn. (6), the directions of $e_{\pm}$ vary, but the orthonormality relations (7) remain fixed. On the other hand, a rescaling of the of the null vectors: $\{e, e_*\} \rightarrow \{\lambda e, \lambda^{-1}e_*\}$ does not affect any of the relations (1) to (5). Thus, we see that our definition of the null basis fixes directions but not scale, whereas the orthonormal basis has fixed scale but arbitrary direction. It is this difference that makes the null basis more suitable for our purposes.

3   Conformal Split

The *conformal split* was introduced in [2] to relate a Minkowski algebra to a maximal Euclidean subalgebra. The split can be defined in two different ways: additively and multiplicatively.

The *additive split* is defined as a direct sum:

$$\mathcal{R}^{n+1,1} = \mathcal{R}^n \oplus \mathcal{R}^{1,1}. \tag{8}$$

For those who would like to see this expressed in terms of a basis, we introduce an orthonormal basis $\{e_k \,|\, e_j \cdot e_k = \delta_{jk}; \ j, k = 1, 2, \ldots, n\}$ for the Euclidean vector

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space $\mathcal{R}^n$ and we note that the orthogonality conditions $e_* \cdot e_k = e \cdot e_k = 0$ are equivalent to the condition that the null vectors anticommute with the e_k, that is,

$$e_k e = -e e_k, \qquad e_k e_* = -e_* e_k. \tag{9}$$

Alternatively, the *multiplicative split* is defined as a direct product:

$$\mathcal{R}_{n+1,1} = \mathcal{R}_n \otimes \mathcal{R}_{1,1}. \tag{10}$$

Here $\mathcal{R}^n$ is actually a space of trivectors with a common bivector factor E. It is related to the vector space $\mathcal{R}^n$ by

$$\mathcal{R}^n = \mathcal{R}^n E \cong \mathcal{R}^n \tag{11}$$

and it generates the Euclidean algebra $\mathcal{R}_n = \mathcal{G}(\mathcal{R}^n)$. It has the basis

$$\{\mathbf{e}_k = e_k E = E e_k\} = \text{trivectors in } \mathcal{R}^3_{n+1,1}. \tag{12}$$

We still have the orthonormality conditions $\mathbf{e}_j \cdot \mathbf{e}_k = e_j \cdot e_k = \delta_{jk}$. However, in contrast to the e_k, the $\mathbf{e}_k$ commute with the null vectors, that is,

$$\mathbf{e}_k e = e \, \mathbf{e}_k, \qquad \mathbf{e}_k e_* = e_* \mathbf{e}_k. \tag{13}$$

This is one very good reason for preferring the multiplicative split over the additive split. The latter was employed in [1], but we will stick with the former.

The multiplicative split $\mathcal{R}_{4,1} = \mathcal{R}_3 \otimes \mathcal{R}_{1,1}$ has significant applications to computational geometry, robotics, computer vision, crytallography and molecular geometry. At a more sophisticated level, the split $\mathcal{R}_{4,2} = \mathcal{R}_{3,1} \otimes \mathcal{R}_{1,1}$ defines a conformal split of spacetime with potential applications to twistor theory and cosmological models in gauge gravity.

4   Models of Euclidean Space

We can model $\mathcal{E}^n$ as a set of points with algebraic properties. A standard way to do that is to identify each Euclidean point with a vector $\mathbf{x}$ in $\mathcal{R}^n$, as expressed by the isomorphism

$$\mathcal{E}^n \cong \mathcal{R}^n. \tag{14}$$

We call this the **inhomogeneous model** of $\mathcal{E}^n$, because the origin $\mathbf{0}$ is a distinguished point in $\mathcal{R}^n$, although all points in $\mathcal{E}^n$ are supposed to be identical.

To eliminate that drawback, we represent points in $\mathcal{E}^n$ by vectors $x \in \mathcal{R}^{n+1,1}$ and, to eliminate the extra degrees of freedom, we suppose that each point lies in the null cone

$$\mathcal{N}^{n+1} = \{x \,|\, x^2 = 0\} \tag{15}$$

and on the hyperplane

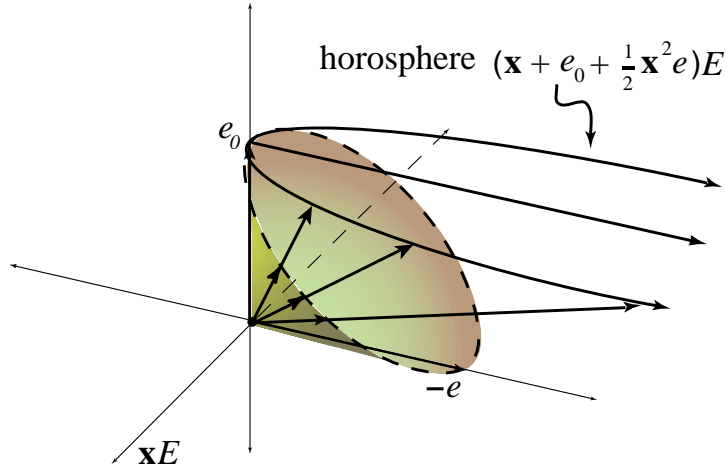$$\mathcal{P}^{n+1}(e, e_*) = \{x \,|\, e \cdot x = 1\}, \tag{16}$$

horosphere $(\mathbf{x} + e_0 + \frac{1}{2}\mathbf{x}^2 e)E$

e_0

$-e$

$\mathbf{x}E$

Fig. 2

where

$$e \cdot (x - e_*) = 0 \quad \Longleftrightarrow \quad e \cdot x = 1 \tag{17}$$

tells us that the plane passes through the point e_*. The intersection of these two surfaces is the *horosphere*:

$$\mathcal{N}_e^{n+1} = \mathcal{N}^{n+1} \cap \mathcal{P}^{n+1}(e, e_*). \tag{18}$$

Thus, we have the isomorphisms

$$\mathcal{R}^n \cong \mathcal{E}^n \cong \mathcal{N}_e^{n+1}. \tag{19}$$

We call the horosphere (Fig. 2) the **homogeneous model** of $\mathcal{E}^n$. It was first constructed by F. A. Wachter (1792–1817), but, as will become apparent, it is only by formulating it in terms of geometric algebra that it becomes a practical tool for computational geometry. To prove the isomorphism (19), we employ a conformal split to relate each homogeneous point $x \in \mathcal{N}_e^{n+1}$ to a unique inhomogeneous point $\mathbf{x} = x \wedge E \in \mathcal{R}^n$.

The conformal split proceeds as follows:

$$x = xE^2 = (x \wedge E + x \cdot E)E,$$

and the constraints on x imply

$$(x \cdot E)E = (x \cdot (e \wedge e_*))E = (\tfrac{1}{2}\mathbf{x}^2 e + e_*)E = -\tfrac{1}{2}\mathbf{x}^2 e + e_*.$$

Whence we obtain an explicit expression for the conformal split:

$$x = (\mathbf{x} + \tfrac{1}{2}\mathbf{x}^2 e + e_*)E = \mathbf{x}E - \tfrac{1}{2}\mathbf{x}^2 e + e_* . \tag{20}$$

From this we calculate

$$x \cdot y = \langle\, xEEy \,\rangle = \mathbf{x} \cdot \mathbf{y} - \tfrac{1}{2}(\mathbf{x}^2 + \mathbf{y}^2)\, e \cdot e_*.$$

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Therefore, the inner product

$$x \cdot y = -\tfrac{1}{2}(\mathbf{x} - \mathbf{y})^2 = -\tfrac{1}{2}(\text{Euclidean distance})^2 \qquad (21)$$

specifies an intrinsic relation among points in $\mathcal{E}^n$.

Setting $\mathbf{x} = 0$ in (20), we see that e_* represents the origin in $\mathcal{R}^n$. From

$$\frac{x}{x \cdot e_*} = \frac{-2}{\mathbf{x}^2}\, \mathbf{x}^2 \left(\frac{1}{\mathbf{x}} + \tfrac{1}{2}e + \frac{1}{\mathbf{x}^2}\, e_* \right) E \xrightarrow[\mathbf{x}^2 \to \infty]{} e,$$

we conclude that e represents a point at infinity.

To facilitate work with the homogenous model, we define I as the *unit pseudoscalar* for $\mathcal{R}^{n+1,1}$, and note the properties

$$|I|^2 = -II^\dagger = 1, \qquad I^\dagger = -I^{-1}, \qquad (22)$$

and $I^\dagger = I$ for $n = 2, 3$. The *dual* of a multivector A in $\mathcal{R}_{n+1,1}$ is defined by

$$\widetilde{A} \equiv AI^{-1} = -AI^\dagger \qquad (23)$$

Therefore, $\widetilde{E} = EI^{-1}$ is the *pseudoscalar* for $\mathcal{R}^n$.

5 Lines and Planes

Grassmann sought to identify the outer product $a \wedge b$ with the line determined by two points a and b [3]. However, he succeeded in doing that only in projective geometry [4]. The homogeneous model enables us to see why he failed to do it in Euclidean geometry: In the Euclidean case, it takes three points to determine a line, and one of them is the point e at infinity. Grassmann could not discover that because he did not have null vectors in his algebraic system.

With the conformal split, we can show that $a \wedge b \wedge e$ can be interpreted as a *line segment* in $\mathcal{E}^n$ or *line* through points a, b, e. The length of the segment is given by its square:

$$(a \wedge b \wedge e)^2 = (a - b)^2 = -2a \cdot b = (\text{length})^2. \qquad (24)$$

Similarly, the outer product $a \wedge b \wedge c \wedge e$ represents a *plane segment* or *plane* in $\mathcal{E}^n$, and the area of the segment is given by its square:

$$(a \wedge b \wedge c \wedge e)^2 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a \cdot b & a \cdot c \\ 1 & b \cdot a & 0 & b \cdot c \\ 1 & c \cdot a & c \cdot b & 0 \end{vmatrix} = 4(\text{area})^2. \qquad (25)$$

This is known, in a slightly different form, as the Cayley-Menger determinant. Cayley discovered it in 1841 and nearly a century later Menger [5] used it in a formulation of Euclidean geometry with interpoint distance as primitive. Dress and Havel [6] recognized its relation to Geometric Algebra.

Using eqn. (20), we can expand the geometric product of points a and b:

$$\begin{aligned} ab &= (aE)(Eb) = (\mathbf{a} + \tfrac{1}{2}\mathbf{a}^2 e + e_*)(\mathbf{b} - \tfrac{1}{2}\mathbf{b}^2 e - e_*) \\ &= -\tfrac{1}{2}(\mathbf{a} - \mathbf{b})^2 + \mathbf{a} \wedge \mathbf{b} + \tfrac{1}{2}(\mathbf{a}^2\mathbf{b} - \mathbf{b}^2\mathbf{a})e + (\mathbf{b} - \mathbf{a})e_* - \tfrac{1}{2}(\mathbf{a}^2 - \mathbf{b}^2)E. \end{aligned}$$

(26)

In expanding (26) we have used the relation $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$, which applies because $\mathbf{a}$ and $\mathbf{b}$ can be interpreted as vectors in $\mathcal{R}_n$, though they are trivectors in $\mathcal{R}_{n+1,1}$. In other words, we have *regraded* the elements of the subalgebra $\mathcal{R}_n$ to conform to our interpretation of $\mathcal{R}^n$ as an inhomogeneous model of $\mathcal{E}^n$. The use of boldface type should avoid confusion between the two different versions of outer product: $\mathbf{a} \wedge \mathbf{b}$ and $a \wedge b$.

The first term on the right side of (26) is recognized as the inner product $a \cdot b$, while the remaining terms make up $a \wedge b$. The profusion of terms in (26) is indicative of the extensive information inherent in the simple product ab. It is similar to the complexity of a spacetime split in physics [7].

From (26) we derive the projective split of a *line* (or *line segment*) through points a, b, e:

$$e \wedge a \wedge b = \mathbf{a} \wedge \mathbf{b}e + (\mathbf{b} - \mathbf{a}).$$

(27)

The coefficients on the right side of (27) will be recognized as the Plücker coordinates for a line with tangent $\mathbf{b} - \mathbf{a}$ and moment $\mathbf{a} \wedge (\mathbf{b} - \mathbf{a}) = \mathbf{a} \wedge \mathbf{b}$, as depicted in Fig. 3. Similarly, the projective split for a *plane* (or *plane segment*) is given by

$$e \wedge a \wedge b \wedge c = -\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}e + (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})E,$$

(28)

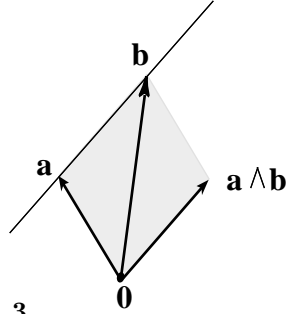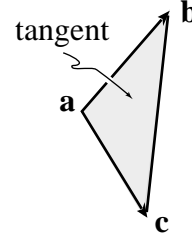where the coefficients are Plücker coordinates for a plane, as depicted in Fig. 4.

Fig. 3   Fig. 4

6 Spheres and Hyperplanes

A *sphere* in $\mathcal{E}^n$ with radius ρ and center p is represented by a vector s in $\mathcal{R}^{n+1,1}$ with positive signature, where

$$\frac{s^2}{(s \cdot e)^2} = \rho^2, \qquad p = \frac{s}{s \cdot e} - \tfrac{1}{2}\rho^2 e.$$

(29)

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It is readily verified that $p^2 = 0$, so p is a homogeneous point. The constraint $s \cdot e = 1$ determines s uniquely and simplifies (29) to

$$s^2 = \rho^2 > 0, \quad p = s - \tfrac{1}{2}\rho^2 e. \tag{30}$$

As depicted in Fig. 5, the equation for the sphere is

$$x \cdot s = 0. \tag{31}$$

This is the equation for a hyperplane through the origin in $\mathcal{R}^{n+1,1}$, although only the vectors satisfying $x^2 = 0$ count as homogeneous points.

The conformal split gives us

$$s = \mathbf{p}\,E + \tfrac{1}{2}(\rho^2 - \mathbf{p}^2)e + e_*. \tag{32}$$

This helps us ascertain that the equation for a circle can be expressed in the alternative forms:

$$x \cdot p = -\tfrac{1}{2}\rho^2 \quad \Longrightarrow \quad \rho^2 = (\mathbf{x} - \mathbf{p})^2. \tag{33}$$
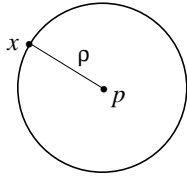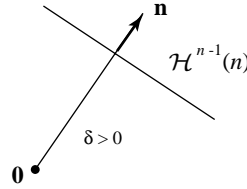

Fig. 5   **Fig. 6**

Like a sphere, a *hyperplane* in $\mathcal{E}^n$ can be represented by a single vector n of positive signature. The vector can be normalized to unity, but it necessarily satisfies

$$n \cdot e = 0. \tag{34}$$

As depicted in Fig. 6, its conformal split has the form

$$n = \mathbf{n}\,E - \delta e, \tag{35}$$

where $n^2 = \mathbf{n}^2 = 1$.

The homogeneous model of $\mathcal{E}^n$ represents *all* spheres and hyperplanes in $\mathcal{R}^n$ as $(n+1)$-dim subspaces of $\mathcal{R}^{n+1,1}$ determined by their normal vectors, as expressed

$$\{x \mid x \cdot s = 0\,,\ s^2 > 0\,,\ s \cdot e \geq 0\,, x^2 = 0;\ x, s \in \mathcal{R}^{n+1,1}\}. \tag{36}$$

For a sphere the normal s satisfies $e \cdot s > 0$, but for a hyperplane it satisfies $e \cdot s = 0$. Thus, a hyperplane is a sphere through the point $e = \infty$.
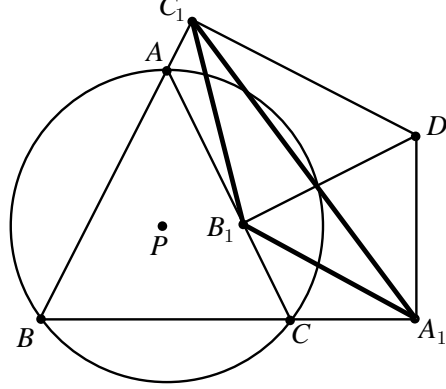
Fig. 7

A sphere determined by $n+1$ points a_0, a_1, a_2, ... a_n is represented by the *tangent form*

$$\tilde{s} \equiv a_0 \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n \neq 0. \tag{37}$$

According to (30), its radius ρ and center p are easily obtained from its dual *normal form*

$$s = -(a_0 \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n)\tilde{}. \tag{38}$$

It follows that the equation for a sphere can be given in the dual forms:

$$x \wedge \tilde{s} = 0 \quad \Longleftrightarrow \quad x \cdot s = 0. \tag{39}$$

These equations apply to a hyperplane as a sphere through ∞ by taking, say $a_0 = e$, to get the dual forms:

$$\tilde{n} = e \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n, \tag{40}$$

$$n = -(e \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n)\tilde{}. \tag{41}$$

Example: Consider Simson's construction shown in Fig. 7. Given a triangle with vertices A, B, C and a point D in a Euclidean plane. Perpendiculars are dropped from D to the three sides of the triangle, intersecting them at points A_1, B_1, C_1.

The circumcircle of triangle $e \wedge A \wedge B \wedge C$ is $\tilde{s} = A \wedge B \wedge C$, so we can obtain its radius from

$$\rho^2 = \left(\frac{s}{s \cdot e}\right)^2 = \frac{\tilde{s}^\dagger \tilde{s}}{(\tilde{s} \wedge e)^2} = \frac{(C \wedge B \wedge A) \cdot (A \wedge B \wedge C)}{(e \wedge A \wedge B \wedge C)^2}. \tag{42}$$

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The following identity can be derived:

$$e \wedge A_1 \wedge B_1 \wedge C_1 = \frac{A \wedge B \wedge C \wedge D}{2\rho^2}. \tag{43}$$

It follows that $A \wedge B \wedge C \wedge D = 0$ if and only if $e \wedge A_1 \wedge B_1 \wedge C_1 = 0$. In other words, D lies on the circumcircle if and only if A_1, B_1, C_1 are collinear. This is Simson's Theorem.

7 Conformal and Euclidean Groups

Orthogonal transformations on Minkowski space are called Lorentz transformations. Any Lorentz Transformation $\underline{G}$ can be expressed in the canonical form

$$\underline{G}(x) = \epsilon\, G\, x\, G^{-1} = \sigma x', \tag{44}$$

where G is a versor with parity $\epsilon = \pm 1$. G is the versor representation of $\underline{G}$, usually called the spin representation if $\epsilon = +1$.

Lorentz transformations leave the null cone $x^2 = x'^2 = 0$ invariant. However, the condition $e \cdot x' = e \cdot x = 1$ is not Lorentz invariant, so a point-dependent scale factor σ has been introduced into (44) to compensate for that.

The Lorentz group on $\mathcal{R}^{n+1,1}$ is isomorphic to the conformal group on $\mathcal{R}^n$, and the two groups are related by the conformal split

$$G\epsilon[\mathbf{x} + \tfrac{1}{2}\mathbf{x}^2 e + e_*]EG^{-1} = \sigma[\mathbf{x}' + \tfrac{1}{2}(\mathbf{x}')^2 e + e_*]E, \tag{45}$$

where

$$\mathbf{x}' = g(\mathbf{x}) \tag{46}$$

is a conformal transformation on $\mathcal{R}^n$.

The great advantage of the versor representation is that it reduces the composition of conformal transformations to versor multiplication, as expressed by the correspondence

$$g_3(\mathbf{x}) = g_2[g_1(\mathbf{x})] \qquad \Longleftrightarrow \qquad G_3 = G_2 G_1.$$

Every versor G can be expressed as the product of non-null vectors, as expressed by $G = s_k \ldots s_2\, s_1$. A vector factor may represent either a hyperplane or a sphere in $\mathcal{R}^n$, as explained in the preceding section.
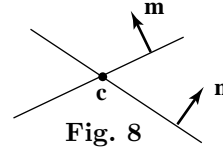
Reflection in the (hyper)plane specified by a vector n, has the simple form:

$$\underline{n}(x) = -nxn^{-1} = x', \tag{47}$$

with $\sigma = 1$. Rotations and translations can be generated multiplicatively from reflections. Thus, reflections in two planes m and n that intersect at a point c (Fig. 8), generate a **rotation** around the line of intersection, as specified by

$$G = mn = (\mathbf{m}E - e\mathbf{m} \cdot \mathbf{c})(\mathbf{n}E - e\mathbf{n} \cdot \mathbf{c})$$

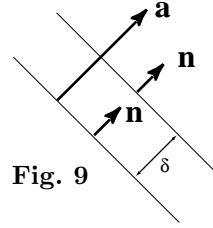$$= \mathbf{mn} - e(\mathbf{m} \wedge \mathbf{n}) \cdot \mathbf{c}$$

$$= R - e(R \times \mathbf{c}).$$

Fig. 8

A **translation** by reflection in parallel planes m, n is specified by

$$G = mn = (\mathbf{m}E - e\delta)(\mathbf{n}E + 0)$$

$$= 1 + \tfrac{1}{2}\mathbf{a}e = T_{\mathbf{a}},$$

where $\mathbf{a} = 2\mathbf{n}\delta$, as shown in Fig. 9.

Fig. 9

The group of rigid displacements on $\mathcal{E}^3$ is called the (special) Euclidean group $SE(3)$. Each group element $\underline{D}$ can be expressed in the form

$$\underline{D}(x) = DxD^{-1}, \tag{48}$$

where the displacement versor $D = T_{\mathbf{a}}R$ specifies a rotation around an axis with direction $\mathbf{n} = R\mathbf{n}R^{\dagger}$ through the origin, followed by a translation $T_{\mathbf{a}} = 1 + \tfrac{1}{2}e\mathbf{a}$.

According to **Chasles' Theorem:** Any rigid displacement can be expressed as a *screw displacement*. This can be proved by finding a point $\mathbf{b}$ on the screw axis so that

$$D = T_{\mathbf{a}_{\parallel}}T_{\mathbf{a}_{\perp}}R = T_{\mathbf{a}_{\parallel}}R_{\mathbf{b}} \tag{49}$$

where $\mathbf{a}_{\parallel} = (\mathbf{a} \cdot \mathbf{n})\mathbf{n}$, and

$$R_{\mathbf{b}} = R + e\,\mathbf{b} \times R, \tag{50}$$

is a rotation that leaves $\mathbf{b}$ fixed. Equation (49) can be solved directly for

$$\mathbf{b} = \mathbf{a}_{\perp}(1 - R^{-2})^{-1} = \tfrac{1}{2}\mathbf{a}_{\perp}\frac{1 - R^2}{1 - \langle R^2 \rangle}. \tag{51}$$

This illustrates the computational power of geometric algebra.

The displacement versor can be put in the *screw* form

$$D = e^{\frac{1}{2}S}, \tag{52}$$

where

$$S = -i\mathbf{m} + e\mathbf{n}, \tag{53}$$

where $i = \widetilde{E}$ is the pseudoscalar for E^3, and S is called a screw. The screws compose $se(3)$, the Lie algebra of $SE(3)$. It is a bivector algebra, closed under the commutator product:

$$S_1 \times S_2 = \tfrac{1}{2}(S_1 S_2 - S_2 S_1). \tag{54}$$

All the elements of screw theory are natural consequences of geometric algebra! Each screw has a unique decomposition into a null and a non-null part:

$$S_k = -i\mathbf{m}_k + e\mathbf{n}_k. \tag{55}$$

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The geometric product of two screws has the decomposition

$$S_1 S_2 = S_1 \cdot S_2 + S_1 \times S_2 + S_1 \wedge S_2. \tag{56}$$

Under a rigid displacement $\underline{U}$, the transformation of a screw is given by

$$S_k' = \underline{U} S_k = U S_k U^{-1} = Ad_U \, S_k. \tag{57}$$

This is the "adjoint representation" of $SE(3)$, as indicated by the notation on the right. The transformation (57) preserves the geometric product:

$$S_1' S_2' = \underline{U}(S_1 S_2) = U(S_1 \cdot S_2 + S_1 \times S_2 + S_1 \wedge S_2)U^{-1}$$

The invariants

$$\underline{U}e = e\,, \qquad \underline{U}i = i \tag{58}$$

imply the invariants:

$$\underline{U}(S_1 \wedge S_2) = S_1 \wedge S_2 = -ie(\mathbf{m}_1 \cdot \mathbf{n}_2 + \mathbf{m}_2 \cdot \mathbf{n}_1), \tag{59}$$

$$S_1' \cdot S_2' = S_1 \cdot S_2 = -\mathbf{m}_1 \cdot \mathbf{m}_2. \tag{60}$$

The latter invariant may be recognized as the Killing Form for $se(3)$.

It is convenient to introduce the notion of a *coscrew* (Ball's reciprocal screw) defined by

$$S_k^* \equiv \langle\, S_k i e_* \,\rangle_2 = \tfrac{1}{2}(S_k i e_* + i e_* S_k) = i\mathbf{n}_k + \mathbf{m}_k e_*. \tag{61}$$

Then the invariant (57) can be written in the scalar-valued form:

$$S_1^* \cdot S_2 = S_2^* \cdot S_1 = \langle\, (S_1 \wedge S_2) i e_* \,\rangle = \mathbf{m}_1 \cdot \mathbf{n}_2 + \mathbf{m}_2 \cdot \mathbf{n}_1. \tag{62}$$

For a single screw we get the *pitch* invariant:

$$h = \tfrac{1}{2}\frac{S^* \cdot S}{S \cdot S} = \mathbf{n} \cdot \mathbf{m}^{-1} \tag{63}$$

8   Screw Mechanics

Screw theory with geometric algebra enables us to combine the rotational and translational equations of motion for a rigid body into a single equation. The kinematics of a body point

$$x = D x_0 D^{-1} \tag{64}$$

is completely characterized by the displacement spinor $D = D(t)$, which obeys the kinematical equation

$$\dot{D} = \tfrac{1}{2}VD \tag{65}$$

with

$$V = -i\boldsymbol{\omega} + \mathbf{v}e, \tag{66}$$

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where $\boldsymbol{\omega}$ is the angular velocity of the body and we can take $\mathbf{v}$ to be its center-of-mass velocity. It follows that $\dot{x} = V \cdot x$ and $\dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x} + \mathbf{v}$.

A *comomentum* P is defined for the body by

$$P = \underline{M}\, V = i\underline{I}\,\boldsymbol{\omega} + m\mathbf{v}e_* = i\boldsymbol{\ell} + \mathbf{p}e_*. \tag{67}$$

This defines a generalized "mass tensor" $\underline{M}$ in terms of the inertia tensor $\underline{I}$ and the body mass m. According to the transformation equations below, the comomentum is a coscrew.

The *coforce* or *wrench* W acting on a rigid body is defined in terms of the torque $\boldsymbol{\Gamma}$ and net force $\mathbf{f}$ by

$$W = i\boldsymbol{\Gamma} + \mathbf{f}e_*. \tag{68}$$

The dynamical equation for combined rotational and translational motion then takes the compact form:

$$\dot{P} = W, \tag{69}$$

where the overdot indicates a time derivative. An immediate consequence is the conservation law

$$\dot{K} = V \cdot W = \boldsymbol{\omega} \cdot \boldsymbol{\Gamma} + \mathbf{v} \cdot \mathbf{f} \tag{70}$$

for kinetic energy

$$K = \tfrac{1}{2}V \cdot P = \tfrac{1}{2}(\boldsymbol{\omega} \cdot \boldsymbol{\ell} + \mathbf{v} \cdot \mathbf{p}). \tag{71}$$

A change of reference frame, including a shift of base point, is expressed by

$$x \quad \longrightarrow \quad x' = \underline{U}x = UxU^{-1}. \tag{72}$$

We consider here only the case when the spinor U is constant. Then (72) induces the transformations

$$V' = \underline{U}V, \tag{73}$$

$$P = \bar{U}P'. \tag{74}$$

Thus, the transformation of V is *Covariant*, while the transformation of P is *Contravariant*. Their scalar product is the *Invariant*

$$P' \cdot V' = P \cdot V. \tag{75}$$

There is much more about all this in [8], [9] and [10], especially applications. For more screw theory, see [11] and [12].

9   Conclusions

We have seen that the homogeneous model for Euclidean space has at least three major advantages.

I. *Intrinsic properties of $\mathcal{E}^n$ are embedded in the algebraic properties of homogeneous points.* In other words we have

$$\left\{\begin{matrix}\text{Synthetic}\\\text{geometry}\end{matrix}\right\} \xrightarrow[\text{with}]{\text{integrated}} \left\{\begin{matrix}\text{Computational}\\\text{geometry}\end{matrix}\right\}$$

This was **Grassmann's great goal**, and he would surely be pleased to know that it has finally been achieved, although the path has not been straightforward.

II. **All** *spheres and hyperplanes in* $\mathcal{E}^n$ *are uniquely represented by vectors in* $\mathcal{R}^{n+1,1}$. This unifies and simplifies the treatment of spheres and hyperplanes, especially with respect to duality properties.

III. $\qquad \left\{\begin{matrix}\text{Conformal group}\\\text{on } \mathcal{E}^n \text{ (or } \mathcal{R}^n)\end{matrix}\right\} \quad \cong \quad \left\{\begin{matrix}\text{Lorentz group}\\\text{on } \mathcal{R}^{n+1,1}\end{matrix}\right\}$

$$\underset{2}{\cong} \quad \left\{\begin{matrix}\text{Versor group}\\\text{in } \mathcal{R}_{n+1,1}\end{matrix}\right\}$$

This isomorphism linearizes the conformal group and reduces composition of conformal transformations to versor multiplication.

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