New Tools for Computational Geometry and rejuvenation of Screw Theory

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Abstract: *Conformal Geometric Algebraic* (CGA) provides ideal mathematical tools for construction, analysis and integration of classical *Euclidean, Inversive & Projective Geometries*, with practical applications to computer science, engineering and physics. This paper is a comprehensive introduction to a CGA tool kit. Synthetic statements in classical geometry translate directly to coordinate-free algebraic forms. Invariant and covariant methods are coordinated by *conformal splits*, which are readily related to the literature using methods of matrix algebra, biquaternions and screw theory. Designs for a complete system of powerful tools for the mechanics of linked rigid bodies are presented.

I. Introduction

Euclidean geometry supplies essential conceptual underpinnings for physics and engineering. It was recognized and reported only recently that *conformal geometric algebra* (CGA) provides an ideal algebraic arena for all aspects of Euclidean geometry, from theoretical formulation and analysis to practical design and computation [1]. I am pleased to say that response to that announcement has been rapid and enthusiastic, with extensive applications ranging from computer science and robotics to crystallography reported in these proceedings and elsewhere.

My purpose here is to set forth the central ideas and results of this "conceptual revolution" as a convenient summary for practitioners and an outline for beginners. For historical context and perspective on the relevant scientific literature, I comment on where the ideas have come from and on opportunities for further development. I take this opportunity to make minor changes and corrections to my previous accounts [1, 2], as well as to clarify and emphasize important points that have been generally overlooked. In particular, I recommend special attention to the different advantages and roles of invariant and covariant approaches to Euclidean geometry and to the prospects for developing and applying Screw Theory.

The adaptation of CGA to serve the purposes of Euclidean geometry is *a fundamental problem in the design of mathematics*. Its objective is a mathematical system that facilitates geometric modeling and analysis, optimizes computational efficiency and incorporates all aspects of rigid body mechanics. Mathematical invention is most effective when its purpose is clear.

II. Universal Geometric Algebra

The geometric concept of vector as a directed number has a long historical development culminating in the invention of *geometric algebra* [3]. To define it we begin with the standard notion of a *real vector space* $\mathbb{R}^{r,s} = \{a, b, c, \ldots\}$ with *dimension* $r + s = n$. On reflection one can see that the concepts of vector addition and scalar multiplication introduced in this way are insufficient to characterize relative directions among vectors. That deficiency is rectified by introducing an associative *geometric product* defined by the simple rule:

$$a^2 = \pm |a|^2 \tag{1}$$

where the real number $|a| \geq 0$ is the *magnitude* of the vector *a*, and the sign is its *signature*. The vector is said to be *null* if $a^2 = |a|^2 = 0$. The vector space $\mathbb{R}^{r,s}$ is presumed to have nondegenerate signature $\{r, s\}$, where *r/s* are maximal subspaces of vectors with *positive/negative* signature. Of course, the signature is not defined without the geometric product, which specifies a multiplicative relation between vectors and scalars. Finally, from the vector space $\mathbb{R}^{r,s}$, the geometric product generates the *real geometric algebra* (GA) $\mathbb{G}^{r,s} = \mathbb{G}(\mathbb{R}^{r,s})$ with elements $\{A, G, M \ldots\}$ called *multivectors*.

Though this completes the definition of GA as an algebraic system, it is only the beginning for *development of GA as a mathematical language*. Development proceeds by creating a system of definitions and theorems that facilitate algebraic encoding and analysis of geometric concepts. A systematic research program to do precisely that was inaugurated in [4] and greatly extended in [5]. I am pleased to say that many others have joined me in this enterprise, including [6, 7, 8] to name only three. These references suffice to show that GA has a broader and deeper range of applications than any other mathematical system, including matrix algebra.

Let me briefly review the basic definitions and theorems needed for most applications of GA: For a pair of vectors, a symmetric *inner product* $a \cdot b$ and antisymmetric *outer product* $a \wedge b$ can be defined implicitly by

$$ab = a \cdot b + a \wedge b \qquad \text{and} \qquad ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b\,. \tag{2}$$

It is easy to prove that $a \cdot b$ is scalar-valued, while the quantity $a \wedge b$, called a *bivector* or 2-vector, is a new algebraic entity that can be interpreted geometrically as an oriented area.

The antisymmetric outer product can be generalized iteratively to define *k-vectors* by

$$a \wedge A_k \equiv \tfrac{1}{2}\left(aA_k + (-1)^k A_k a\right), \tag{3}$$

which generates a *(k+1)-vector* from *k-vector* A_k. It follows that the outer product of *k* vectors is the completely antisymmetric part of their geometric product:

$$a_1 \wedge a_2 \wedge \ldots \wedge a_k = \langle a_1 a_2 \ldots a_k \rangle_k\,, \tag{4}$$

where the angle bracket means *k-vector part* and *k* is its *grade.* This product vanishes if and only if the vectors are linearly dependent. Consequently, the maximal grade for nonzero *k-vectors* is $k = n$. It follows that every multivector *A* can be expanded into its *k*-vector parts and the entire algebra can be decomposed into *k*-vector subspaces:

$$\mathbb{G}^{r,s} = \sum_{k=0}^{n} \mathbb{G}_k^{r,s} = \left\{ A = \sum_{k=0}^{n} \langle A \rangle_k \right\} \tag{5}$$

This is called a *grading* of the algebra. Note that the grading is generated from primitive elements, the vectors or 1-vectors in this case, with the scalars regarded as elements with grade 0. As seen below, alternative gradings are appropriate for geometric subalgebras.

The inner product can also be generalized, leading to the very useful formula

$$a \cdot (a_1 \wedge a_2 \wedge \ldots \wedge a_k) = \sum_{j=1}^{k} (-1)^{j+1} a \cdot a_j (a_1 \wedge \ldots \wedge \breve{a}_j \wedge \ldots \wedge a_k)\,, \tag{6}$$

where $\breve{a}_j$ indicates a missing factor in the outer product. This formula shows that the inner product is a grade-lowering operator, while (3) shows that the outer product is grade-raising.

Reversing the order of multiplication is called *reversion*, as expressed by

$$(a_1 a_2 \dots a_k)^{\sim} \equiv a_k \dots a_2 a_1 , \qquad \text{whence} \qquad (a_1 \wedge a_2 \wedge \dots \wedge a_k)^{\sim} = a_k \wedge \dots \wedge a_2 \wedge a_1 , \tag{7}$$

and the *reverse* of an arbitrary multivector is defined by

$$\tilde{A} = \sum_{k=0}^{n} \langle \tilde{A} \rangle_k \equiv \sum_{k=0}^{n} (-1)^{k(k-1)/2} \langle A \rangle_k \tag{8}$$

Similarly, *(space) inversion* (regarded as the result of reversing the sign of all vectors in geometric products) is defined by

$$A^{\#} \equiv \sum_{k=0}^{n} (-1)^k \langle A \rangle_k = \langle A \rangle_+ - \langle A \rangle_- , \tag{9}$$

where $\langle A \rangle_\pm$ are the parts of $A = \langle A \rangle_+ + \langle A \rangle_-$ with *even/odd parity* respectively.

The unit *n-vector* (or *pseudoscalar*) is so important that it is given a special symbol and defined (up to a sign) by the properties

$$\mathrm{I} = \langle \mathrm{I} \rangle_n , \qquad \mathrm{I}\tilde{\mathrm{I}} = (-1)^s , \qquad a \wedge \mathrm{I} = 0 \text{ for every vector } a. \tag{10}$$

Every multivector A has a *dual* defined by $A^* \equiv A\mathrm{I}^{-1}$. This leads to the basic theorem relating inner and outer products by duality:

$$a \cdot A^* = a \cdot (A\mathrm{I}^{-1}) = (a \wedge A)\mathrm{I}^{-1} = (a \wedge A)^* . \tag{11}$$

The algebraic system described to this point may be referred to as *universal GA*, because of its broad applicability. That distinguishes it from specialized versions of GA, such as the *spacetime algebra* [4, 6] tailored to the geometry of spacetime.

Names and nomenclature are of great importance in science and mathematics, as they can suppress or reveal deep conceptual distinctions. In this regard, it is important to mark crucial conceptual differences between *Geometric Algebra* (GA) and *Clifford Algebra* (CA) that are often overlooked in the literature. Though they have a common root in the work of W. K. Clifford, CA has been cultivated by mathematicians up to present times as one among many formal algebraic systems with little attention to its geometric meaning. In contrast, the systematic development of GA as a universal geometric language is a more recent development that is still underway. Since Clifford himself proposed the name *Geometric Algebra*, I believe he would be embarrassed to have it named after him; for he was well aware of its universal geometric import, and he attributed to Hermann Grassmann [9] the chief role in its creation. Indeed, Clifford's development of the algebra hardly progressed beyond the rudiments, though it is clear that he had deep insights into the geometric and algebraic issues [10]. No doubt the history of Geometric Algebra would have been quite different if not for Clifford's tragic early death.

Here are some important differences in viewpoint between CA and GA. CA is typically defined with complex numbers instead of the reals as scalars, whereas GA contends that this obscures geometric meaning without providing greater generality. CA is often defined as an ideal in tensor algebra, whereas GA defines tensors as multilinear functions of vector variables [5]. CA is often characterized as the "algebra of a quadratic form," with $a \cdot b$ interpreted as a metric tensor. In contrast, GA contends that the inner product should be regarded as a *contraction* (or grade-lowering operation) as originally conceived by Grassmann. Any metric tensor can then be defined as a scalar-valued bilinear function $g(a,b) = a \cdot \underline{g}(b)$, where $\underline{g}$ is a linear operator, with

$a \cdot b$ as the most important special case. Many further differences between GA and CA are obvious in applications.

III. Group Theory with Geometric Algebra

A multivector G that can be expressed as a geometric product $G = n_k \ldots n_2 n_1$ of non-null vectors is called a *versor*. Obviously it has a multiplicative inverse $G^{-1} = n_1^{-1} n_2^{-1} \ldots n_k^{-1}$ and even or odd parity given by $G^{\#} = (-1)^k G$.

The multiplicative group of all versors in $\mathbb{G}^{r,s}$ is the *Pin group*:

$$\mathrm{Pin}(r,\ s) = \left\{ G : GG^{-1} = 1 \right\}, \tag{12}$$

within which the subgroup of all versors with even parity is the *Spin group*:

$$\mathrm{Spin}(r,\ s) = \left\{ \mathrm{G}\text{: } \mathrm{G}^{\#} = G \right\}. \tag{13}$$

The *orthogonal group* on $\mathbb{R}^{r,s}$ is the group of all *isometries*, that is, linear transformations $\underline{G}$ that preserve the magnitude of each vector a. In GA it has the elegant representation:

$$\mathrm{O}(r,\ s) = \left\{ \underline{G} : \underline{G}(a) = G^{\#} a G^{-1} \right\} \tag{14}$$

The versors in $\mathrm{Pin}(r,\ s)$ are thus generators of the orthogonal group. The versors of $\mathrm{Spin}(r,\ s)$ generate the *special orthogonal group* $\mathrm{SO}(r,\ s)$, a subgroup of $\mathrm{O}(r,\ s)$ sometimes called the rotation group.

This GA approach to isometries has considerable advantages over matrix representations: First, it is completely coordinate-free. Second, it reduces group composition of isometries $\underline{G}_2 \underline{G}_1 = \underline{G}_3$ to simple multiplication of versors $G_2 G_1 = G_3$. At the same time, it establishes direct connection between the orthogonal group and its *covering* by the pin group. Third, it facilitates reduction of isometries to their irreducible elements, namely, reflection in a hyperplane determined by its vector normal:

$$\underline{G}_i(a) = n_i^{\#} a n_i^{-1} = -n_i a n_i^{-1} \tag{15}$$

It is a simple matter then to prove the important

> *Cartan-Dieudonné Theorem*: Every isometry of $\mathbb{R}^{r,s}$ can be reduced to at most $n = r + s$ reflections in hyperplanes.

As usual, credit for the theorem could probably be more fairly attributed to others, most notably to Lipschitz (1880), who pioneered the approach to isometries presented here. The best practice may be to give it a descriptive name such as "*Isometry Reduction Theorem.*"

Evidently the GA approach can be profitably extended to the whole of group representation theory [5]. The *classical groups* have been treated in [11]. We will be most interested below in the *conformal group* C(*r, s*), which has a representation in GA specified by the isomorphism

$$\mathrm{C}(r,\ s) \cong \mathrm{O}(r+1,\ s+1) \tag{16}$$

This representation is so useful that we shall refer to $\mathbb{G}^{r+1,s+1}$ as *Conformal Geometric Algebra* (CGA).

This completes our summary of universal GA and its relation to group theory. In the following we concentrate on practical applications to Euclidean geometry, knowing full well that our results are readily generalized to spaces of arbitrary dimension and signature.

IV. Euclidean Geometry with Conformal GA

The conformal model of Euclidean 3-space $\mathbb{E}^3$ is embedded in the CGA $\mathbb{G}^{4,1} = \mathbb{G}(\mathbb{R}^{4,1})$ as follows: First, we identify Euclidean points with vectors in the null cone

$$\mathbb{N}^{4,1} \equiv \left\{ x \in \mathbb{R}^{4,1} : \quad x^2 = 0 \right\}. \tag{17}$$

Next, we reduce the remaining degrees of freedom from four to three by choosing a point at infinity $e \equiv x_\infty$ and normalizing all points to the hyperplane $\left\{ x : \ e \cdot x = -1, \ e^2 = 0 \right\}$. Thus, we have

$$\mathbb{E}^3 \cong \mathbb{N}_e^{4,1} \equiv \left\{ x \in \mathbb{R}^{4,1} : \quad x^2 = 0, \quad x \cdot e = -1 \right\}. \tag{18}$$

Finally, we confirm this as a model of Euclidean space by verifying that

$$| x_2 - x_1 |^2 = (x_2 - x_1)^2 = -2 x_2 \cdot x_1 \tag{19}$$

correctly determines the *Euclidean distance* $| x_2 - x_1 |$ between any two point. The argument is completed below.

The amazing fact about this embedding of $\mathbb{E}^3$ in CGA is that it automatically imbues all elements of $\mathbb{G}^{4,1}$ with rich geometric meaning, and thereby facilitates formulation, analysis, and computation in all aspects of Euclidean geometry. It has two major advantages:

First, it unites the conceptual advantages of classical synthetic geometry with the analytic power of algebra in providing direct algebraic representations of basic geometric objects and their properties.

Second, it enlists the apparatus of the conformal versor groups for multiplicative, coordinate-free representation of Euclidean symmetries and transformations. Specifically, the invariance group of the Euclidean metric (19) is the *Euclidean group* $\mathrm{E}(3) = \left\{ \underline{G} \right\}$, defined as a subgroup of the conformal group C(3, 0) ≅ O(4, 1) by the constraint

$$\underline{G}(e) = G^{\#} e G^{-1} = e \tag{20}$$

This group includes reflections. Its restriction to rigid displacements by requiring $G^{\#} = G$ is the *Special Euclidean group* SE(3).

The rest of this paper is an elaboration of these two points with specific recommendations for notation, representation and method. The subject is young and fluid, so the setting of standards for practice is still open. Of course, I cannot cover everything. For further details and explanation I refer the serious student to [7], which provides the most thorough exposition of CGA to date, with due emphasis on geometric visualization. Comparison with the present account shows where I think that exposition can be improved.

In CGA the basic **Geometric Objects** $\{O = C, L, S, P\}$ of 3D Euclidean geometry can be defined as follows:

A **Circle** C is determined by three points:

$$C = x_1 \wedge x_2 \wedge x_3 \,. \tag{21}$$

A **Line** L is a circle through the point at infinity:

$$L = x_1 \wedge x_2 \wedge e \,. \tag{22}$$

A **Sphere** S is determined by four points:

$$S = x_1 \wedge x_2 \wedge x_3 \wedge x_4 \tag{23}$$

A **Plane** P is a sphere through the point at infinity:

$$P = x_1 \wedge x_2 \wedge x_3 \wedge e \tag{24}$$

A **Point** x *lies on* object O if and only if

$$x \wedge O = 0 . \qquad (25)$$

Note the distinction between a geometric object O (defined algebraically) and the set of points $\mathbb{O}$ it determines, as expressed by

$$\mathbb{L}\text{ine} \equiv \{x \mid x \wedge L = 0\} \qquad \mathbb{P}\text{lane} \equiv \{x \mid x \wedge P = 0\} \qquad (26)$$

In this respect we follow Euclid in introducing points and lines as distinct objects with properties specified by a system of axioms. The idea of defining a line as a set of points emerged in the 19th century with "containment" replacing the geometric concept of "incidence" as the basic relation between points and lines. From our perspective, the limitations of that idea are clear, so we are prepared to use the concepts of set theory but not to confuse them with concepts of geometry.

Of course our concept of geometric object goes beyond Euclid's, most notably in assigning to each an *orientation* (algebraic sign) and a *weight* or *magnitude* (e.g. *length, area, volume*). Thus, interchanging the product of points in (21-24) reverses the sign, hence orientation, of the objects.

For many purposes the *dual representation* for a geometric object $O^* = O\mathrm{I}^{-1}$ is most convenient. From the duality of inner an outer products (11) it follows that the intersection with a point (25) is then expressed by

$$x \cdot O^* = 0 . \qquad (27)$$

For a plane, the dual $P^* = P\mathrm{I}^{-1} = n$ is a vector *normal* to the plane (note the use of lower case letters for vectors). The equation $x \cdot n = 0$ has the familiar form of an equation for a plane through the origin of a vector space, but in this case it applies to any plane in $\mathbb{E}^3$. For the normal n specifies a location as well as an orientation for the plane. Moreover, the separation of $\mathbb{E}^3$ into disjoint subsets can be neatly expressed by the inequality $x \cdot n > 0$ for points *in front of* the plane, and $x \cdot n < 0$ for points *behind* the plane.

The *intersection* of two planes $P_1 = n_1\mathrm{I}$ and $P_2 = n_2\mathrm{I}$ is a line specified by

$$P_1^* \cdot P_2 = n_1 \cdot P_2 = n_1 \cdot (n_2\mathrm{I}) = (n_1 \wedge n_2)\mathrm{I} . \qquad (28)$$

Obviously, this vanishes if the planes are parallel. Moreover, as will be evident later, with the normalization $n_1^2 = n_2^2 = 1$, the magnitude $| n_1 \wedge n_2 |$ is the *sine* of the dihedral angle between the intersecting planes.

Similar expressions for the mutual intersections of lines, planes, circles and spheres are discussed in [7].

V. Invariant Euclidean Geometry

There are two different ways to formulate the equations of spacetime physics: (1) *covariant formulations* expressed with respect to one inertial frame and related to other frames by Lorentz transformations. (2) *invariant formulations* independent of any reference frame choice. Experts prefer to work with invariants, because they are invariably simpler than covariants. However, beginners are usually introduced to a covariant approach, mainly because of educational tradition.

In precise analogy, there are *covariant formulations* of Euclidean geometry that depend on designating an arbitrary point as origin, and *invariant formulations* that do not. The conformal model supports both approaches, so we should examine their respective advantages and how they are related.

The characterization of geometric objects in the preceding section is already an invariant formulation. Let us consider it more closely for extension to an invariant treatment of any topic in Euclidean geometry. We have seen that CGA supplies sufficient algebraic structure to define basic geometric objects. Now note that the structure of CGA suggests a somewhat different approach to geometric primitives than the classical one.

The primitive algebraic objects are vectors. In CGA there are four types of vector with distinct geometric meanings:

$$\mathbb{P}\text{oints:} \quad \{x \mid x^2 = 0, x \cdot e = -1\}$$
$$\mathbb{P}\text{lanes:} \quad \{n \mid n^2 > 0, \ n \cdot e = 0\}$$
$$\mathbb{S}\text{pheres:} \quad \{s \mid s^2 = \pm\rho^2, \ s \cdot e = -1\} \tag{29}$$

This suggests that the dual form for a plane $n = P^*$ should be regarded as more fundamental than the 4-vector form in (24). It is also algebraically much more convenient, especially for generating translations, as shown below.

According to (29) there are two types of sphere with radius ρ, corresponding to the two signs in the square of the sphere vector s. The length of the sphere vector is scaled so its square gives the radius directly. The two sphere types are designated as *real* and *imaginary* for positive and negative square respectively. A *real sphere* $s = S^*$ is the dual of the 4-vector sphere S in (23). It is of interest to note that the *center* c of a real sphere can be obtained by a suitably scaled reflection from the point at infinity:

$$c = -\frac{1}{2}ses = -\frac{1}{2}(2e \cdot s - es)s = s + \frac{1}{2}\rho^2 e \tag{30}$$

An easy check verifies that c does indeed have the properties of a point. Moreover, this gives us a natural measure for distance from a point to a sphere:

$$2s \cdot x = 2(c - \frac{1}{2}\rho^2 e) \cdot x = \rho^2 - |x - c|^2 \tag{31}$$

Thus, the point x is *inside, on* or *outside* the sphere when $s \cdot x$ is *positive, zero* or *negative* respectively. The order is reversed by changing the sign (orientation) of s.

Note that the reflection (30) has been defined with respect to the radius of the sphere instead of unity. This kind of reflection is called *inversion in a sphere*. Applied to an arbitrary point, it gives a new point:

$$x' = -\frac{1}{2}sxs\,, \tag{32}$$

and a little algebra reveals the distance inversion

$$(x' - c)^2 = \frac{\rho^4}{(x - c)^2}\,, \tag{33}$$

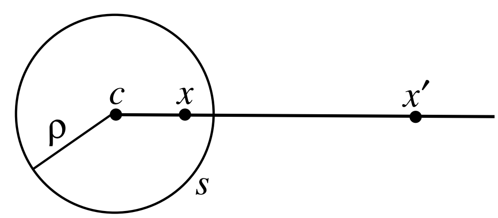

Fig. 1

as illustrated in Fig. 1.
Though sphere inversion does not preserve Euclidean distance, it is a powerful means for geometric design and analysis.

The geometric significance of *imaginary spheres* is more subtle, and the reader is referred to [7] for a discussion. The main point of interest here is that all four vector types in (29) are needed for computational Euclidean geometry. Subject to scaling, these constitute all the vectors in the

CGA $\mathbb{G}^{4,1} = \mathbb{G}(\mathbb{R}^{4,1})$. We can conclude then that CGA supplies precisely the algebraic structure needed for Euclidean geometry without anything superfluous.

Geometry can be regarded as a system of relations among points. Accordingly, the most basic is the relation between two points described by the vector $n_{21} = x_2 - x_1$. This vector is so important that it deserves a name. I like the venerable old term *chord,* especially when it is relating points in a geometric figure or physical object. Of course, it serves as a *displacement vector* in other contexts. However, it has another geometric property unique to CGA; it is the *perpendicular bisector* of the interval between the two points. The fact that it is the normal for a plane is confirmed immediately by $n_{21} \cdot e = 0$. The fact that is the bisecting plane is confirmed from the equation $n_{21} \cdot x = 0$, which, as illustrated in Fig. 2, implies (using (19)) that

$$| x - x_1 |^2 = -2x \cdot x_1 = | x - x_2 |^2 .$$

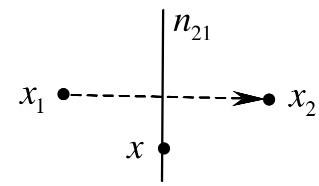

Fig.2.

The chords of a *triangle* are especially significant, for they determine the basic properties of the Euclidean metric:

$$| x_i - x_j |^2 = n_{ij}^2 \geq 0 .$$

The chords are related by the triangle equation

$$n_{21} + n_{32} + n_{13} = 0 .$$

This gives us immediately the familiar *law of cosines*:

$$n_{21}^2 + n_{32}^2 + 2n_{32} \cdot n_{21} = n_{13}^2 ,$$

which determines the basic triangle inequalities for Euclidean distances.

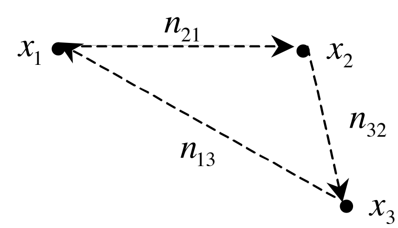

Fig.3.

Versor products among the chords generate all the reflection and rotation symmetries of a triangle and, consequently, values for all the vertex angles. For example, the versor $n_{32}n_{21}$ is "complementary" to a rotation "about the vertex x_2," and vanishing of its scalar part reduces the law of cosines to the Pythagorean theorem. Note that the chord n_{ij} generates a reflection that takes point x_i to x_j or vise versa. Hence, the versor product of successive chords generates a walk of reflections along any sequence of points, which may return to the initial point if the path is closed, as in a walk around a triangle.

There is much more to be derived from the invariant approach to Euclidean geometry. For example, according to (21) and (24) the outer product of three vertices in a triangle determines its *circumcircle* and the plane in which it lies. Of course, all this applies to 2D as well as 3D geometry. It would interesting to work out what insights and simplifications it brings to the great theorems of classical geometry, such as the *nine circle theorem*. Indeed, the results may even have practical value in applications to mechanical engineering, as we see in later sections.

Finally, to complete our discussion of two point geometry we note that the sum

$$s_{21} = x_2 + x_1 = c_{21} + \frac{1}{2}\rho_{21}^2 e \tag{34}$$

is a sphere with center c_{21} and poles at the two points. However, in contrast to the real sphere (30), it is an imaginary sphere. Its role in Euclidean geometry remains to be worked out.

VI. Projective Geometry

Projective geometry is useful in many applications – in computer vision, for example -- but its methodology stands apart from the rest of mathematics. To make the conceptual assets of projective geometry readily available, we need to incorporate them into the algebraic design of CGA. As Dieudonné has famously declared, projective geometry is nothing but linear algebra. Accordingly, let us consider a generic, non-singular linear transformation that leaves the point at infinity invariant (up to a scale factor σ at least):

$$\underline{\mathrm{f}}:\ x \mapsto x' = \underline{\mathrm{f}}(x) \qquad \text{with} \qquad \underline{\mathrm{f}}(e) = \sigma e\,. \tag{35}$$

The trouble with this is that it need not preserve the null property of points, so we have

$$\underline{\mathrm{f}}:\ x^2 = 0 \ \mapsto\ \left[\underline{\mathrm{f}}(x)\right]^2 = \underline{\mathrm{f}}(x)\cdot\underline{\mathrm{f}}(x) = x\cdot\overline{\mathrm{f}}\underline{\mathrm{f}}(x) \neq 0$$

(Note: the underbar notation $\underline{\mathrm{f}}$ denotes a linear operator while the overbar $\overline{\mathrm{f}}$ denotes its adjoint.) To solve this problem, Anthony Lasenby [12] has proposed that we extend the notion of points to include planes regarded as boundary points at ∞. Thus, we extend our model of Euclidean space to include two kinds of points:

Interior points: $\{x \mid x^2 = 0,\quad x\cdot e = -1\}$

Boundary points: $\{n \mid n^2 = 1,\quad n\cdot e = 0\}$,

where the boundary points, like the interior points, are normalized to make them unique. The set of boundary points can thus be regarded as a $\mathbb{P}$lane (of directions) at ∞. Indeed, each boundary point can be regarded as the intersection of parallel lines at ∞, as parallel lines have a common direction. This is an old idea dating back to Kepler.

Now, projective geometry suppresses metrical notions of scale while maintaining the geometric concept of *incidence*, as expressed by equation (28). As that equation requires use of the pseudoscalar and duality, we must extend our notion of projective transformations to accommodate them. Happily, GA provides a natural way to do precisely that.

A great advantage of GA is that it enables natural extension of a linear transformation on vectors to the entire algebra. This extension is called an *outermorphism* [13, 5, 6, 7] because it preserves the outer product (hence grade), as expressed by

$$\underline{\mathrm{f}}(x\wedge M) = \underline{\mathrm{f}}(x)\wedge\underline{\mathrm{f}}(M)\,. \tag{36}$$

It follows that the pseudoscalar is an eigenblade of the outermorphism, with the determinant as its eigenvalue:

$$\underline{\mathrm{f}}(\mathrm{I}) = (\det\underline{\mathrm{f}})\,\mathrm{I}\,. \tag{37}$$

This prepares us for the fundamental theorem [13]:

$$(\det\underline{\mathrm{f}})\,\underline{\mathrm{f}}(A^*\cdot B) = \underline{\mathrm{f}}(A)^*\cdot\underline{\mathrm{f}}(B)\ , \tag{38}$$

which can be made to look more elegant by absorbing $(\det\underline{\mathrm{f}})$ in A^* defined as the dual with respect to the transformed pseudoscalar (37). This theorem could fairly be called the *Incidence Theorem*, because it expresses the fact that outermorphisms preserve the incidence property (28).

This fundamental theorem of linear algebra has been almost totally overlooked in the literature, presumably because it is not so naturally expressed in standard formalisms.

These developments invite us to employ the pseudoscalar to define “complex objects” such as

$$N = x + \mathrm{I}n \ , \tag{39}$$

where x is an interior point and n is a boundary point. As explained in a later section, this object can interpreted as a *point in a plane* if $n \cdot x = 0$ or, equivalently, $N^2 = -1$. All this suggests that we should define *projective transformations* by extending outermorphisms to include duality transformations.

I believe that we now have all the necessary ingredients to incorporate projective geometry smoothly into CGA. There remains the large task of reformulating the classical results of projective geometry. Since modern works have already formulated many of these results in terms of linear algebra [14], the task should be fairly straightforward. I recommend it as a good topic for a doctoral thesis.

VII. Covariant Euclidean Geometry with Conformal Splits

The most widely used model of Euclidean geometry by far is the *vector space model* based on the isomorphism of Euclidean space to a real vector space with Euclidean inner product:

$$\mathbb{E}^3 \cong \mathbb{R}^3 = \{\mathbf{x}\}. \tag{40}$$

The most effective means of exploiting this model is through its geometric algebra:

$$\mathbb{G}^3 = \mathbb{G}(\mathbb{R}^3) = \{\alpha + \mathbf{a} + i\mathbf{b} + i\beta\}, \tag{41}$$

where i is the unit right-handed pseudoscalar, and the geometric product of vectors articulates perfectly with the standard dot and cross products:

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + i\mathbf{a}\times\mathbf{b}. \tag{42}$$

Efficient methods for applying $\mathbb{G}^3$ to any aspect of mechanics are well developed with many innovative features [15]. In particular, details of the quaternion theory of rotations are thoroughly worked out and smoothly articulated with standard vector methods and matrix representations.

These results even articulate smoothly with the arcane literature on applications of complex quaternions to geometry and mechanics. For it is evident in (41) that complex quaternions are isomorphic to multivectors in $\mathbb{G}^3$, though practitioners have not realized that their unit imaginary can be interpreted geometrically as a pseudoscalar

Despite all these advantages, the algebra $\mathbb{G}^3$ suffers from the drawback of all vector space models, namely, that the vector space (40) singles out the origin as a preferred point. In other words, it introduces an asymmetry that is not inherent in the concept of Euclidean space. Happily, that can be remedied by embedding the vector space model in the conformal model, or better, by factoring it out of the conformal model. We consider two ways to do that.

The first way is a ***conformal split*** of CGA into a commuting product of subalgebras:

$$\mathbb{G}^{4,1} = \mathbb{G}^3 \otimes \mathbb{G}^{1,1} \tag{43}$$

The split is defined geometrically by choosing one point e_0 as origin and noting that every other point x lies on the bundle of lines through that point. This defines a mapping of points into trivectors:

$$\mathbf{x} \equiv x \wedge e_0 \wedge e\,, \tag{44}$$

which we identify with the vectors in (40). Thus, with a *regrading* of trivectors as vectors, we generate $\mathbb{G}^3$ as a subalgebra of $\mathbb{G}^{4,1}$.

The other subalgebra $\mathbb{G}^{1,1} = \mathbb{G}(\mathbb{R}^{1,1})$ in (43) is generated from the null vectors $\{e_0,\ e\}$. Its pseudoscalar is a bivector of sufficient importance to merit a special symbol:

$$E = e_0 \wedge e\,, \qquad \text{with} \qquad E^2 = (e \cdot e_0)^2 = 1\,. \tag{45}$$

We examine this algebra more fully later on. For now, it suffices to note that its content, though not its structure, depends on the arbitrary choice of the origin point e_0. It is *covariant* in the sense that it changes with a change of origin. I have dubbed it *conformal split,* because it is deeply analogous to the *spacetime split* [6, 4], which is so useful in spacetime physics. The spacetime split is generated by selecting a timelike vector rather than a null vector as here. Otherwise, the structure and utility of the splits are quite comparable.

The nature of the conformal split may be clarified by examining a basis for $\mathbb{R}^{4,1}$:

$$\{e, e_0, e_1, e_2, e_3\} \quad \text{with} \quad e_j \cdot e_k = \delta_{jk} \ \text{ for } j,\, k = 1, 2, 3 \quad \text{and} \quad e \cdot e_k = 0 = e_0 \cdot e_k\,. \tag{46}$$

This generates a basis for $\mathbb{R}^3$:

$$\left\{\boldsymbol{\sigma}_k = e_k \wedge e_0 \wedge e = e_k(e_0 \wedge e) = e_k E = E e_k\right\} \tag{47}$$

and a pseudoscalar

$$i = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = (e_1E)(e_2E)(e_3E) = e_1e_2e_3E = \mathrm{I} \tag{48}$$

Thus, the pseudoscalar for $\mathbb{G}^3$ is identical to the pseudoscalar for $\mathbb{G}^{4,1}$. It is an invariant of the conformal split!

An alternative to the conformal split is the *additive split*:

$$\mathbb{G}^{4,1} = \mathbb{G}(\mathbb{R}_+{}^3 \oplus \mathbb{R}^{1,1}) \equiv \mathbb{G}_+{}^3 \oplus \mathbb{G}^{1,1}\,, \tag{49}$$

defined by choosing $\{e_1, e_2, e_3\}$ from (46) as a basis for $\mathbb{R}_+{}^3$. Unlike the basis (47), the basis in this case is not algebraically associated with lines through a point, and the pseudoscalar $\mathrm{I}_3 \equiv e_1e_2e_3 = \mathrm{I}E$ is not an invariant. Furthermore, the $\boldsymbol{\sigma}_k$ commute with e while the e_k do not. Consequently, the additive split is not as convenient as the conformal split. Even so, it has its place, most notably in modeling a rigid body, as we shall see.

To demonstrate the felicity of the conformal split for relating invariant forms for geometric objects to standard vector space forms, results for the most basic geometric objects (point, line, plane) are given here.

The mapping (44) of point x to vector $\mathbf{x} = x \wedge E$ can be inverted. The slickest way to do that is to use the geometric product thus:

$$xE = x \wedge E + x \cdot E \qquad \text{with} \qquad x \cdot E = x \cdot (e_0 \wedge e) = (x \cdot e_0)e + e_0 \tag{50}$$

Multiplying the first equation by its reverse, we get

$$0 = (x \wedge E)^2 - (x \cdot E)^2\,; \qquad \text{whence} \qquad \mathbf{x}^2 = (x \cdot E)^2 = -2x \cdot e_0\,. \tag{51}$$

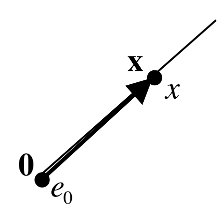

Fig. 4

Inserting this back into (50), we solve to get

$$x = (\mathbf{x} - \frac{1}{2}\mathbf{x}^2 e + e_0)E = E(\mathbf{x} + \frac{1}{2}\mathbf{x}^2 e - e_0) = \mathbf{x}E + \frac{1}{2}\mathbf{x}^2 e + e_0 \,. \tag{52}$$

This can be regarded as the conformal split of point x with respect to point e_0. Illustrations of the two representations for points are superimposed in Fig. 4, which may be misleading because the points are related by projection.

The conformal split of a line L through points x and a gives

$$L = x \wedge a \wedge e = \mathbf{x} \wedge \mathbf{a}e + (\mathbf{a} - \mathbf{x}) = (\mathbf{d}e + 1)\mathbf{n} \tag{53}$$

Note that this represents the line in terms of its *Plücker coordinates,* which consists of a vector, bivector pair for the line tangent and moment with respect to the origin (Fig. 5):

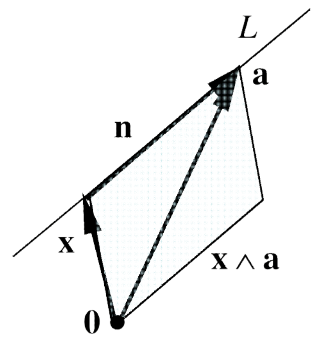

Fig. 5

$$\underline{\text{tangent}}: \quad \mathbf{n} = \mathbf{a} - \mathbf{x} \tag{54}$$

$$\underline{\text{moment}}: \quad \mathbf{x} \wedge \mathbf{a} = \mathbf{x} \wedge (\mathbf{a} - \mathbf{x}) = \mathbf{dn} \tag{55}$$

The directed distance from origin to line is given by the *directance*:

$$\mathbf{d} = (\mathbf{x} \wedge \mathbf{a})\mathbf{n}^{-1} = (\mathbf{x} \wedge \mathbf{n})\mathbf{n}^{-1} = \mathbf{x} - (\mathbf{x} \cdot \mathbf{n}^{-1})\mathbf{n} \,. \tag{56}$$

The conformal split of a plane P through points x, a, b gives

$$P = x \wedge a \wedge b \wedge e = \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{b}e + (\mathbf{a} - \mathbf{x}) \wedge (\mathbf{b} - \mathbf{x})E \,. \tag{57}$$

Its *Plücker coordinates* consists of the bivector–trivector pair (Fig. 6):

$$\underline{\text{tangent}}: \quad (\mathbf{a} - \mathbf{x}) \wedge (\mathbf{b} - \mathbf{x}) = \mathbf{x} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{x} = i\mathbf{n} \tag{58}$$

$$\underline{\text{moment}}: \quad \begin{aligned} \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{b} &= \mathbf{x} \wedge [(\mathbf{a} - \mathbf{x}) \wedge (\mathbf{b} - \mathbf{x})] \\ &= \mathbf{x} \wedge (i\mathbf{n}) = i(\mathbf{x} \cdot \mathbf{n}) \end{aligned} \tag{59}$$

The dual form for the plane is:

$$P = i(\mathbf{x} \cdot \mathbf{n}e + \mathbf{n}E) = in \tag{60}$$

More explicitly, split of the plane normal n (Fig. 2) gives us

$$\begin{aligned} n = x_2 - x_1 &= (\mathbf{x}_2 - \mathbf{x}_1)E + \frac{1}{2}(\mathbf{x}_2{}^2 - \mathbf{x}_1{}^2)e \\ &= (\mathbf{x}_2 - \mathbf{x}_1)E + \frac{1}{2}(\mathbf{x}_2 + \mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1)e = \mathbf{n}E + \mathbf{c} \cdot \mathbf{n}e \end{aligned} \tag{61}$$

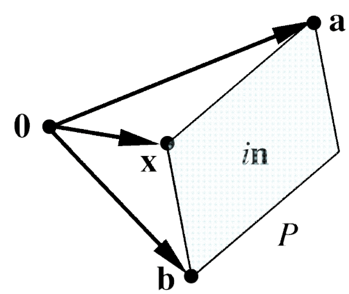

Fig. 6

The invariant forms for geometric objects are obviously much simpler than the split forms. Therefore it is preferable to work with invariant forms directly. However, the split forms are essential for relating results to the literature, so we will be using them for that purpose below.

VIII. Rigid Displacements

From equation (20) it follows that every *rigid displacement* $\underline{D}$ is a linear transformation of the form:

$$\underline{D}: \; x \;\mapsto\; x' = \underline{D}(x) = Dx\tilde{D} \,, \tag{62}$$

where its generator D is a versor of even parity that commutes with the point at infinity and is normalized to unity; that is,

$$D^{\#} = D\,, \qquad\qquad De = eD\,, \qquad\qquad DD^{-1} = D\tilde{D} = 1\,. \tag{63}$$

With respect to any chosen point e_0, the displacement can decomposed into a rotation $\underline{R}$ followed by a translation $\underline{T}$ or vice versa. This defines a *conformal split* of the displacement as follows:

$$\underline{D} = \underline{T}\,\underline{R} = \underline{R}'\underline{T}\,, \tag{64}$$

where the rotations satisfy

$$\underline{R}(e_0) = Re_0\tilde{R} = e_0\,, \qquad\qquad \underline{R}'(e_0') = R'e_0'\tilde{R}' = e_0'\,, \tag{65}$$

and the translation

$$\underline{T}(e_0) = Te_0\tilde{T} = e_0' = \underline{D}(e_0)\,, \tag{66}$$

is determined solely by the endpoints e_0 and e_0'. For given $\underline{D}$ and any choice of e_0, the translation can be computed and the rotation is determined by $R = \tilde{T}D$.

As all displacements can be generated from reflections in planes, let us consider the various possibilities along with their conformal splits.

Reflection in a plane with unit normal n and $e \cdot n = 0$ is specified by

$$x' = \underline{n}(x) = -nxn = x - 2x \cdot nn \tag{67}$$

If the plane passes through a point c,
we have $c \cdot n = 0$ and the conformal split

$$n = \mathbf{n}E + \mathbf{c}\cdot\mathbf{n}e \qquad \text{with} \qquad x \cdot n = \mathbf{n}\cdot(\mathbf{x}-\mathbf{c})\,. \tag{68}$$

Whence,

$$\mathbf{x}' = \mathbf{x} - 2(\mathbf{x}-\mathbf{c})\cdot\mathbf{n}\,\mathbf{n}\,, \tag{69}$$

as shown in Fig. 7.

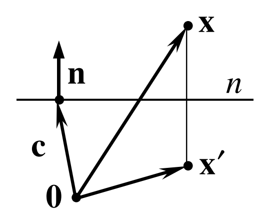

Fig. 7

Rotation by planes n and m intersecting through a point c (Fig. 8) is generated by

$$\begin{aligned} R_c &= mn = (\mathbf{m}E + \mathbf{m}\cdot\mathbf{c}e)(\mathbf{n}E + \mathbf{n}\cdot\mathbf{c}e) \\ &= \mathbf{mn} + e(\mathbf{m}\wedge\mathbf{n})\cdot\mathbf{c} = R + e(R\times\mathbf{c}) = T_{\mathbf{c}}^{-1}RT_{\mathbf{c}}, \end{aligned} \tag{70}$$

where $T_{\mathbf{c}}$ generates the translation from origin e_0 to c, $R = \mathbf{mn}$, and we have used *the commutator product*, defined by

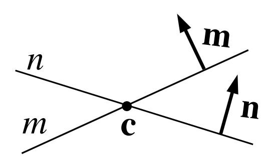

Fig. 8

$$A \times B \equiv \frac{1}{2}(AB - BA)\,. \tag{71}$$

Note that R_c in (70) can be identified with R' in (65) if $c = e_0'$ and $T_{\mathbf{c}}^{-1} = T$ in (66).

Translation through parallel planes n and m is generated by

$$T_{\mathbf{a}} = mn = (\mathbf{n}E + 0)(\mathbf{n}E + \delta e) = 1 + \frac{1}{2}\mathbf{a}e\,, \tag{72}$$

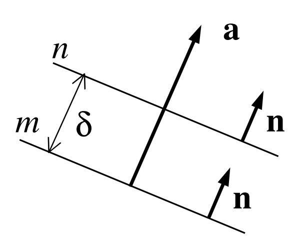

Fig. 9

where $\mathbf{a} = 2\mathbf{n}\delta$, and, without loss of generality, one plane is presumed to pass through the origin (Fig. 9).

Now we can use (72) and (61) to evaluate $T = T_{\mathbf{a}}$ in (66), with the result

$$a = e_0' - e_0 = \mathbf{a} + \frac{1}{2}\mathbf{a}^2 e\,. \tag{73}$$

IX. Framing a Rigid Body

The *position* and *attitude* of a rigid body in space is uniquely determined by specifying the positions of four points, say $\{x, x_1\ x_2\ x_3\}$, embedded in the body. Identifying position with the *base point x,* the attitude can be represented by the *body frame* $\{e_k = x_k - x\}$, as illustrated in Fig. 10. And it is most convenient to othonormalize the body frame, so $e_j \cdot e_k = \delta_{jk}$.

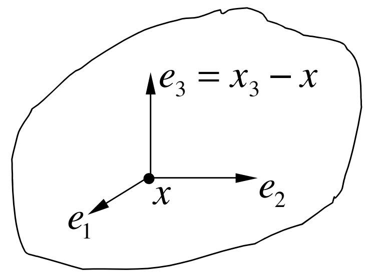

Fig. 10

The body frame represents attitude by a set of three vectors. A promising alternative representation in terms of a single geometric object has been proposed by Selig [16] following ideas of Engels. He defines a *Flag* geometrically as *a point on a line in a plane*. CGA gives it the elegant algebraic form:

$$F = x + L + P = x + \mathrm{I}Q\,, \tag{74}$$

where line L and plane P are defined by

$$L = x \wedge x_1 \wedge e = x \wedge (x_1 - x) \wedge e = x \wedge e_1 \wedge e = \mathrm{I}n_2 n_3,$$
$$P = x \wedge x_1 \wedge x_2 \wedge e = x \wedge e_1 \wedge e_2 \wedge e = e_2 \wedge L = \mathrm{I}n_3,$$

and their combined dual forms are given by

$$Q = n_3 + n_2 n_3 = (1 + n_2) n_3\,, \tag{75}$$

where $n_j \cdot n_k = \delta_{jk}$. As the n_k are the normals for intersecting planes, they are represented in Fig. 11 by arrows extending symmetrically to each side of the base point. Of course, the fact that the base point lies on the intersection of line and plane is expressed by

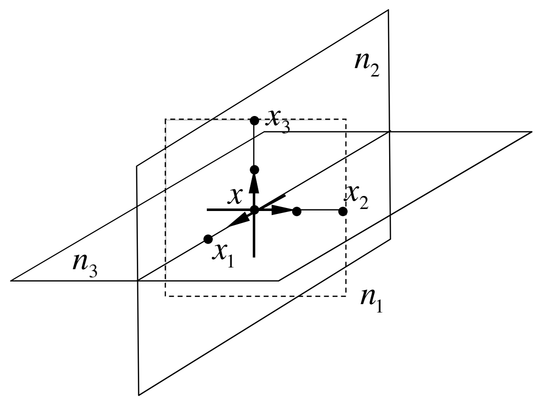

Fig. 11

$$x \wedge (L + P) = x \wedge (\mathrm{I}Q) = 0\,, \quad \text{or dually by} \quad x \cdot Q = 0\,. \tag{76}$$

Lasenby [12] arrived at Q in a different way, and, noting that $Q^2 = 0$, he identified it with the mysterious *absolute conic* of projective geometry.

As a related connection to projective geometry, note that the "complex vector" $N = x + \mathrm{I}n$ introduced in (39) is a flag without the line component. There are many other possibilities to explore, such as introducing vectors representing spheres instead of planes.

It seems simplest to work with dual forms for line and plane. This suggests that we consider a *dual flag* defined by

$$F^* = x + Q = x + n_3 + n_2 n_3\,, \quad \text{with} \quad x \cdot F^* = 0\,. \tag{77}$$

This looks unsymmetrical, as the plane n_1 is not explicitly represented, though it is determined indirectly by the intersection of the base point with the other planes. Here is a more symmetrical representation for the rigid body:

$$Q' \equiv n_3 + n_2 n_3 + n_1 n_2 n_3, \qquad \text{with} \qquad x \cdot Q' = 0\,. \tag{78}$$

Note that this representation is a graded sum of nested subspaces with pseudoscalar $\mathrm{I}_3 = n_1 n_2 n_3$ intrinsic to the body — a practical instance of the additive split (49).

X. Rigid Body Kinematics

Motion of a rigid body is a one parameter family of displacements, which we describe by a time dependent versor function $D = D(t)$. As illustrated in Fig. 12, this determines the evolution of body points from some reference positions e_k to instantaneous positions

$$x_k(t) = De_kD^{-1} = De_k\tilde{D} \tag{79}$$

From the versor character of D it can be proved that its derivative must satisfy

$$\dot{D} = \tfrac{1}{2}VD \quad \text{and} \quad \dot{D}^{-1} = -\tfrac{1}{2}D^{-1}V\,, \tag{80}$$

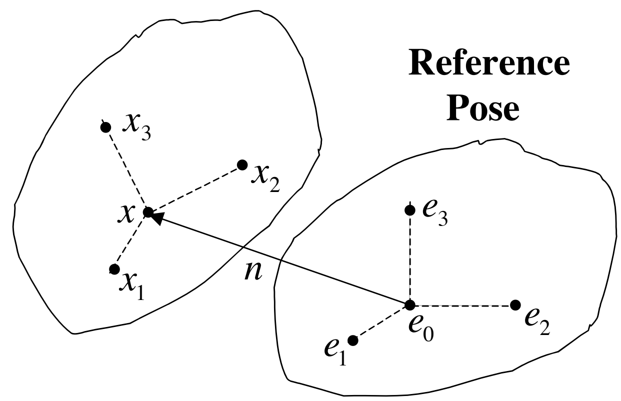

Fig. 12

where the *velocity* $V = V(t)$ is a bivector, as expressed algebraically by $\tilde{V} = -V = \langle V \rangle_2$.

Using (80) to differentiate (79), we get equations of motion for the body points:

$$\dot{x}_k = V \cdot x_k\,. \tag{81}$$

However, there is no need to integrate this system of three equations, as the body motion is completely determined by integrating the *displacement equation* (80). Indeed, the equation of motion for D is independent of any designation of specific body points, although selection of a base point is necessary to separate rotational and translational components of the motion.

To decompose the (generalized) velocity V into rotational and translational parts, we introduce a conformal split defined by:

$$D = RT\,, \qquad De_0D^{-1} = Te_0T^{-1} = e_0 + n\,, \qquad T = 1 + \tfrac{1}{2}ne\,. \tag{82}$$

Derivatives of rotation and translation versors have the form

$$\dot{R} = -\tfrac{1}{2}i\boldsymbol{\omega}R \qquad \dot{T} = \tfrac{1}{2}\dot{n}e = \tfrac{1}{2}\dot{x}e = \tfrac{1}{2}\dot{x}eT = \tfrac{1}{2}\dot{\mathbf{x}}eT\,, \tag{83}$$

where $\boldsymbol{\omega}$ is the *rotational velocity* of the body and $\dot{\mathbf{x}} = \dot{x} \wedge E$. Hence

$$\dot{D} = \dot{R}T + R\dot{T} = \tfrac{1}{2}(-i\boldsymbol{\omega} + Re\dot{\mathbf{x}}R^{-1})RT = \tfrac{1}{2}VD\,,$$

so the velocity has the split form

$$V = -i\boldsymbol{\omega} + e\mathbf{v}\,, \qquad \text{with} \qquad \mathbf{v} = R\dot{\mathbf{x}}R^{-1}\,. \tag{84}$$

One can write $\mathbf{v} = \dot{\mathbf{x}}$ by adopting the instantaneous initial condition $R(t) = 1$ (as done implicitly in most references on kinematics), although that complicates further differentiation of $\mathbf{v}$ if needed. (See the Section on rotating systems in [15] for further discussion of this point.) Also, the negative sign for rotational velocity in (83) and (84) is dictated by the convention that the rotation is right handed around the oriented $\boldsymbol{\omega}$ axis [15].

Now consider the effect of shifting the initial base point from the origin e_0 to

$$e_0' = T_0 e_0 \tilde{T}_0 = e_0 + r_0, \qquad \text{where} \qquad T_0 = 1 + \frac{1}{2}\mathbf{r}_0 e. \tag{85}$$

This determines a shift in base point trajectory to

$$x' = D e_0' \tilde{D} = D' e_0 \tilde{D}', \qquad . \tag{86}$$

where

$$D' = D T_0 = T_{\mathbf{r}} D, \qquad \text{with} \qquad T_{\mathbf{r}} = R T_0 \tilde{R} = 1 + \frac{1}{2}\mathbf{r} e. \tag{87}$$

Differentiating, we have

$$\dot{D}' = \dot{T}_{\mathbf{r}} D + T_{\mathbf{r}} \dot{D} = \frac{1}{2} e \dot{\mathbf{r}} T_{\mathbf{r}} D + \frac{1}{2} V D T_0 = \frac{1}{2}(e\dot{\mathbf{r}} + V) D' = \frac{1}{2} V' D'.$$

Thus, we have proved that a shift in base point induces a shift in velocity:

$$V = -i\boldsymbol{\omega} + e\mathbf{v} \quad \mapsto \quad V' = V + e\dot{\mathbf{r}} = -i\boldsymbol{\omega} + e(\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}) \tag{88}$$

This result is the kinematic version of **Chasles' Theorem** [15]. Note that the rotational part is independent of the base point shift.

The base point need not be located within the rigid body, so at a given time the vector $\mathbf{r}$ can be specified freely. In particular, one can specify

$$\boldsymbol{\omega} \cdot \mathbf{r} = 0 \qquad \text{since} \qquad \mathbf{r} = \boldsymbol{\omega}^{-1} \times \mathbf{v} = i(\mathbf{v} \wedge \boldsymbol{\omega}^{-1}) \tag{89}$$

to put V' in the form of a *screw*:

$$V' = -i\boldsymbol{\omega} + eh\boldsymbol{\omega} = \boldsymbol{\omega}(eh - i) \qquad \text{with } \textit{pitch} \qquad h = \mathbf{v} \cdot \boldsymbol{\omega}^{-1} = \mathbf{v} \cdot \boldsymbol{\omega} / \boldsymbol{\omega}^2 . \tag{90}$$

For positive pitch, this velocity generates an infinitesimal translation along the axis of a right-handed rotation.

From (79) and (82) it follows that chords $n_k = e_k - e_0 = T(e_k - e_0)T^{-1}$ are invariant under translations. Hence, the evolution of chords and products of chords is simply a rotation, as described by

$$n_k' = D n_k \tilde{D} = R n_k \tilde{R} \qquad \text{and} \qquad n_j' n_k' = D n_j n_k \tilde{D} = R n_j n_k \tilde{R}. \tag{91}$$

Likewise, evolution of the dual flag Q^* in (79) is described by:

$$Q^* \quad \rightarrow \quad Q^*(t) = D Q^* \tilde{D} = R Q^* \tilde{R}. \tag{92}$$

Note that the form of these rotations is independent of base point, though the value of R is not, as described explicitly by equation (70).

XI. Rigid Body Dynamics

In [15] GA is employed for a completely coordinate-free derivation and analysis of the equations for a rigid body. The results are summarized here for embedding in a more compact and deeper formulation with CGA. Then [15] can be consulted for help with detailed applications.

For a rigid body with total *mass m* and *inertial tensor* $\underline{I}$, the *momentum* **p** and *rotational* (or *angular*) *momentum* **l** are defined by

$$\mathbf{p} = m\mathbf{v} \qquad \text{and} \qquad \mathbf{l} = \underline{I}(\boldsymbol{\omega}), \tag{93}$$

where **v** is the velocity of a freely chosen base point (not necessarily the center of mass), and the inertia tensor depends on the choice of base point, as determined by the *parallel axis theorem* (See [15], where the structure of inertia tensors is discussed in detail).

Translational and rotational motions are then determined (respectively) by *Newton's Force Law* and *Euler's Torque Law*:

$$\dot{\mathbf{p}} = \mathbf{f} = \sum_k \mathbf{f}_k \qquad \text{and} \qquad \dot{\mathbf{l}} = \underline{I}(\dot{\boldsymbol{\omega}}) + \boldsymbol{\omega} \times \underline{I}(\boldsymbol{\omega}) = \boldsymbol{\Gamma} = \sum_k \boldsymbol{\Gamma}_k, \tag{94}$$

where the *net force* **f** is the sum of forces applied to specified body points, and the *net torque* $\boldsymbol{\Gamma}$ is the sum of applied torques.

Now, to combine **p** and **l** into a generalized *comomentum P* that is linearly related to the velocity V in (88), we introduce a generalized *mass operator* $\underline{M}$ defined by

$$P = \underline{M}V = m\mathbf{v}e_0 - i\underline{I}\boldsymbol{\omega} = \mathbf{p}e_0 - i\,\mathbf{l} \tag{95}$$

The appearance of e_0 instead of e in this expression requires some explanation. For the moment it suffices to note that it yields the standard expression for total *kinetic energy*:

$$K \equiv \frac{1}{2}V \cdot \tilde{P} = -\frac{1}{2}V \cdot \underline{M}V = \frac{1}{2}(\boldsymbol{\omega} \cdot \mathbf{l} + \mathbf{v} \cdot \mathbf{p}). \tag{96}$$

Next, using (95) we combine the two conservation laws (94) into a single equation of motion for a rigid body:

$$\dot{P} = W \qquad \text{where} \qquad W = \mathbf{f}e_0 - i\boldsymbol{\Gamma} \tag{97}$$

is called a *wrench* or *coforce*. From this we easily derive the standard expression for *power* driving change in kinetic energy:

$$\dot{K} = V \cdot \tilde{W} = \boldsymbol{\omega} \cdot \boldsymbol{\Gamma} + \mathbf{v} \cdot \mathbf{f}. \tag{98}$$

Finally, we consider the effect of shifting the base point as specified by (85) and (86). That shift induces shifts in the comomentum and applied wrench:
Parallel axis theorem

$$P \quad \mapsto \quad P' = P + i\mathbf{r} \times \mathbf{p} = \mathbf{p}e_0 - i(\mathbf{l} - \mathbf{r} \times \mathbf{p}) \tag{99}$$

$$W \quad \mapsto \quad W' = W + i\,\mathbf{r} \times \mathbf{f} = e_0\mathbf{f} - i(\boldsymbol{\Gamma} - \mathbf{r} \times \mathbf{f}) \tag{100}$$

The comomentum shift expresses the *parallel axis theorem*, while the corresponding shift in torque is sometimes called *Poinsot's theorem.* As a check for consistency with Chasles' theorem (88), we verify shift invariance of the kinetic energy:

$$2K = V' \cdot \tilde{P}' = -(V + e\boldsymbol{\omega} \times \mathbf{r}) \cdot (P + i\mathbf{r} \times \mathbf{p})$$
$$= \boldsymbol{\omega} \cdot (\mathbf{l} - \mathbf{r} \times \mathbf{p}) + (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{p} = \boldsymbol{\omega} \cdot \mathbf{l} + \mathbf{v} \cdot \mathbf{p} = V \cdot \tilde{P}.$$

This completes our transcription of rigid body dynamics into a single invariant equation of motion (97). As indicated by the appearance of e_0 in (95) and (97) and verified by the shift equations (99) and (100), separation of the motion into rotational and translational components is a conformal split that depends on the choice of base point. In applications there is often an optimal choice of base point, such as the point of contact of interacting rigid bodies, as in the rigid body linkages discussed below.

This is a good place to reflect on what makes CGA mechanics so compact and efficient. In writing my mechanics book [15], I noted that momentum is a vector quantity while angular momentum is a bivector quantity, so I combined them by defining a "complex velocity"

$$V = \mathbf{v} + i\boldsymbol{\omega}\,, \qquad (101)$$

with corresponding definitions for complex momentum and force. That led to a composite equation of motion just as compact as (97). However, it was more a curiosity than an advantage, because you had to take it apart to use it. The trouble was, as I fully understood only with the development of CGA, that the complex velocity (101) does not conform to the structure of the Euclidean group. That defect is remedied by the simple expedient of introducing the null element *e* to change the definition of velocity from (101) to (84).

The resulting velocity (84) is a bivector in CGA. To make that explicit, we note that the first term is a bivector because it is the dual of a trivector: $\boldsymbol{i\omega} = \mathrm{I}\omega \wedge E = \mathrm{I} \cdot (\omega \wedge e_0 \wedge e)$, while the second term is the contraction of a trivector by a vector: $e\mathbf{v} = (v \wedge E)e = (v \wedge e_0 \wedge e) \cdot e$. Thus, equation (84) is a generic form for bivectors generating Euclidean displacements. Of course, that fact was implicit already in the definition of *V* in the displacement equation (80). It has been reiterated here to confirm consistency with the conformal split. As we see next, the notion of *V* as bivector generator of the Euclidean group is the foundation of Screw Theory. That is what makes the equation of motion (97) so significant.

XII. Screw Theory

Screw theory was developed in the latter part of the nineteenth century [17] from applications of geometry and mechanics to the design of mechanisms and machines. When formulated within the standard matrix algebra of today the concepts of screw theory seem awkward or even a bit *screwy*! Consequently, applications of screw theory, deep and useful though they be, have remained outside the mainstream of mechanical engineering.

Here we cast screw theory in terms of CGA to secure its rightful place in the Kingdom of Euclidean geometry and facilitate access to its rich literature. To the extent that the conformal model becomes a standard for applications of Euclidean geometry, this will surely promote a rejuvenation of screw theory.

The foundations for screw theory in rigid body mechanics have been laid in the preceding sections. Here we concentrate on explicating the screw concept in relation to displacements. For constant *V*, the displacement equation (80) integrates immediately to the solution

$$D(t) = e^{\frac{1}{2}Vt} = e^{\frac{1}{2}\underline{T}_r V't} = T_r e^{\frac{1}{2}V't} T_r^{-1}, \qquad (102)$$

where (87) to (90) have been used in the form

$$V = \underline{T}_r V' = T_r V' T_r^{-1} = -i T_r \boldsymbol{\omega} T_r^{-1} + eh\boldsymbol{\omega} = -i\boldsymbol{\omega} + e(h\boldsymbol{\omega} + \mathbf{r}\times\boldsymbol{\omega}) = -i\boldsymbol{\omega} + e\mathbf{v} \tag{103}$$

to exhibit the conformal split and shift of base point. The translation versor, which, of course, generates a fixed displacement, also has an exponential form:

$$T_r = 1 + \frac{1}{2}\mathbf{r}e = e^{\frac{1}{2}\mathbf{r}e}, \qquad T_r^{-1} = e^{-\frac{1}{2}\mathbf{r}e} = T_{-r}. \tag{104}$$

As expressed by (103), the rotation rate $\boldsymbol{\omega}$ is invariant under a base point shift. As $\boldsymbol{\omega} = \omega \wedge E$ is a trivector representing a line through the origin, the motion generated by $D(t)$ in (102) is a steady screw motion with constant pitch along that line. The translations in (102) can be understood as translating the line through a given base point to one through the origin, unfolding the screw displacement, and then translating it back to the original base point.

To consolidate our concepts it is helpful to introduce nomenclature that conforms to the screw theory literature as closely as possible. In general, any Euclidean displacement versor can be given the exponential form:

$$D = e^{\frac{1}{2}S}, \qquad \text{where} \qquad S = i\mathbf{m} + e\mathbf{n} \tag{105}$$

The versor D is called a *twistor*, while its generator S is called a *twist* or a *screw.* The term "screw" is often restricted to the case where $\mathbf{n}$ and $\mathbf{m}$ are collinear and $\mathbf{m}^2 = 1$. The line determined by $\mathbf{m}$ is called the *screw axis* or *axode*.

As implicitly shown in (62) and (63), the multiplicative group of twistors is a double covering of the Special Euclidean group:

$$\mathrm{SE}(3) = \{\text{rigid displacements } \underline{D}\} \underset{2}{\cong} \{\text{twistors } D\}. \tag{106}$$

The set of all twists constitutes an algebra of bivectors:

$$se(3) \equiv \textit{Lie algebra} \text{ of } \mathrm{SE}(3) = \{S_k = i\mathbf{m}_k + e\mathbf{n}_k\}. \tag{107}$$

This algebra is closed under the commutator product:

$$S_1 \times S_2 = \frac{1}{2}(S_1S_2 - S_2S_1) = i(\mathbf{m}_2 \times \mathbf{m}_1) + e(\mathbf{n}_2 \times \mathbf{m}_1 - \mathbf{n}_1 \times \mathbf{m}_2). \tag{108}$$

Let us summarize important general properties of this algebra

The representation of a Lie group by action on its Lie algebra is called the adjoint representation [16]. In this case, we have

$$S_k' = \underline{U}(S_k) = US_kU^{-1} = Ad_U\, S_k, \qquad \text{where} \qquad \{\underline{U}\} = \mathrm{SE}(3) \tag{109}$$

This transformation preserves the geometric product $S_1S_2 = S_1 \cdot S_2 + S_1 \times S_2 + S_1 \wedge S_2$; that is,

$$S_1'S_2' = \underline{U}(S_1S_2) = U(S_1 \cdot S_2 + S_1 \times S_2 + S_1 \wedge S_2)U^{-1}. \tag{110}$$

Separating parts of different grade, we see first that the commutator product is *covariant*, as expressed by

$$S_1' \times S_2' = \underline{U}(S_1 \times S_2) = U(S_1 \times S_2)U^{-1}. \tag{111}$$

Second, the scalar part is an obvious invariant:

$$S_1' \cdot S_2' = S_1 \cdot S_2 = -\mathbf{m}_1 \cdot \mathbf{m}_2, \tag{112}$$

known as the *Killing form* for the group. Finally, the remaining term is a pseudoscalar invariant

$$S_1' \wedge S_2' = \underline{U}(S_1 \wedge S_2) = S_1 \wedge S_2 = ie(\mathbf{m}_1 \cdot \mathbf{n}_2 + \mathbf{m}_2 \cdot \mathbf{n}_1), \tag{113}$$

because it is proportional to the *Euclidean pseudoscalar* $\mathrm{I_e} = ie$, which is invariant because *i* and *e* are invariant. This concept and result is unique to CGA, so it merits further discussion.

The pseudoscalar $\mathrm{I_e} = ie$ squares to $\mathrm{I_e^2} = -e^2 = 0$, so it cannot be used for an invertible duality mapping. However, we can define a conjugate pseudoscalar $\mathrm{I_e^*} \equiv ie_0$ so that $\mathrm{I_e^*} \cdot \mathrm{I_e} = \langle \mathrm{I_e^* I_e} \rangle = -e \cdot e_0 = 1$. Then we can express the invariant (113) as a scalar:

$$(S_1 \wedge S_2) \cdot (ie_0) = \mathbf{m}_1 \cdot \mathbf{n}_2 + \mathbf{m}_2 \cdot \mathbf{n}_1 = S_1 \cdot S_2^*, \tag{114}$$

where a *dual screw* = *coscrew* has been defined by

$$S_k^* \equiv S_k \cdot (ie_0) = \langle S_k ie_0 \rangle_2 = -i\,\mathbf{n}_k - e_0 \mathbf{m}_k, \tag{115}$$

This equivalent to the *reciprocal screw* first introduced by Ball [17]. Comparison with (97) shows that *wrenches are coscrews*! Of course, the dual defined here should not be confused with the dual introduced in (11). The asterisk notation has been used for both to emphasize their conceptual commonality.

Finally, we note that for a single screw, the *pitch h* appears as a ratio of the invariants (114) and (112):

$$h = -\frac{1}{2}\frac{S \cdot S^*}{S \cdot S} = \mathbf{n} \cdot \mathbf{m}^{-1}. \tag{116}$$

Screw theory is basically about the generators of displacements. The simplicity of its formulation within CGA belies the richness and complexity of its applications in mechanical engineering, for which the serious student must consult the literature.

As guides to the screw theory literature, I recommend two books. The first book [16] concerns use of modern mathematical concepts and notations, which can be compared to the approach taken here. The second book [18] is by two long-time practitioners of screw theory. Though it is designed as a textbook for the ill-prepared engineering student of today, it provides a mature perspective on current status with an authoritative entrée to the literature.

Screw theory literature going back to the nineteenth century contains many gems that can be recovered by reformulation within CGA. Here is another worthy topic for doctoral research. It requires more than historical study, for many of the gems need polishing — there are unsolved problems to be addressed in the new light of CGA.

To facilitate translations from the literature, relations between CGA and matrix algebra are established next.

XIII. Conformal Split and Matrix Representation

Besides its invariance and incredible compactness, one great advantage of the CGA formulation of rigid body mechanics in the preceding sections is the ease of relating it to alternative formulations by a conformal split. In this section we consider the conformal split in more detail, especially to clarify and facilitate connections to the vast literature on mechanical systems.

As defined in (43), one factor in the conformal split is the geometric algebra $\mathbb{G}^{1,1} = \mathbb{G}(\mathbb{R}^{1,1})$. The vector space $\mathbb{R}^{1,1}$ is sometimes referred to as 2D Minkowski space to emphasize its similarity to 4D Minkowski space $\mathbb{R}^{3,1}$, which is a standard model for spacetime in relativistic physics [4, 6]. It can be generated from a *null basis* $\{e,\ e_0 \mid e^2 = e_0{}^2 = 0,\ e\cdot e_0 = -1\}$ or from an equivalent *orthonormal basis* $\left\{e_\pm = \frac{1}{\sqrt{2}}(\lambda e \mp \lambda^{-1} e_0),\ \lambda \neq 0,\ e_\pm{}^2 = \pm 1\right\}$. Though the orthonormal basis is more familiar to most readers, as we have seen already, the null basis is more significant geometrically. It generates a basis $\{1,\ e,\ e_0,\ E\}$ (depicted in Fig. 13) for the entire algebra $\mathbb{G}^{1,1}$, with the basic properties:

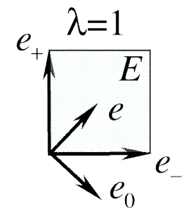

Fig. 13

$$E^2 = 1,\quad e_0 e = E - 1,\quad Ee = -eE = e,\quad e_0 E = -Ee_0 = e_0. \tag{117}$$

These properties have been used many times in previous sections.

The algebra of *dual numbers* $\mathbb{D} = \{\alpha + e\beta\}$ is an important subalgebra of $\mathbb{G}^{1,1}$. However, it was first proposed by Clifford as an extension of the real numbers analogous to complex numbers, with the null unit *e* replacing the imaginary unit *i*. He introduced it as an extension of scalars in quaternions to form what he called biquaternions [10, 16]. We can regard it as an extension of the algebra $\mathbb{G}^3$ to $\mathbb{G}^3 \otimes \mathbb{D}$. Clifford clearly recognized the geometric significance of this extension for incorporating the additivity and commutativity properties of translations. In terms of translation versors, these properties are expressed by

$$T_{\mathbf{a}} T_{\mathbf{b}} = (1 + \frac{1}{2}\mathbf{a}e)(1 + \frac{1}{2}\mathbf{b}e) = 1 + \frac{1}{2}(\mathbf{a} + \mathbf{b})e = T_{\mathbf{a}+\mathbf{b}} = T_{\mathbf{b}} T_{\mathbf{a}}\ . \tag{118}$$

Clifford's biquaternions have been used to represent translations and screws by many authors since. However, we have seen in the preceding section that the dual numbers must be extended to the entire algebra $\mathbb{G}^{1,1}$ to accommodate coscrews and screw invariants. That has been done in the literature primarily by employing matrices in the following way.

The algebra $\mathbb{G}^{1,1}$ is isomorphic to the algebra $\mathbb{M}_2(\mathbb{R})$ of real 2×2 matrices. That is readily established by exhibiting the isomorphism of basis elements:

$$e_+ \simeq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad e_- \simeq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad E \simeq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\quad 1 \simeq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{119}$$

Accordingly, every multivector *M* in $\mathbb{G}^{1,1}$ has a matrix representation $\hat{M}$ explicitly given by

$$M = \tfrac{1}{2}\left[A(1+E) + B(e_+ + e_-) + C(e_+ - e_-) + D(1-E)\right] \simeq \hat{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{120}$$

where the matrix elements are real numbers. This representation is readily generalized by allowing the matrix elements to have values in other algebras, $\mathbb{G}^3$ in particular. Thus, we arrive at the isomorphism:

$$\mathbb{G}^{4,1} = \mathbb{G}^3 \otimes \mathbb{G}^{1,1} \simeq \mathbb{M}_2(\mathbb{G}^3)\,. \tag{121}$$

Properties of this isomorphism are surprisingly rich and have been thoroughly studied in [13]. That enabled a critique of the matrix representation for the conformal group, which contributed to developing the invariant formulation in CGA introduced in [1].

The matrix algebra $\mathbb{M}_2(\mathbb{G}^3)$ has been much used in screw theory with the elements of $\mathbb{G}^3$ interpreted as complex quaternions. More often, it has been used with the elements of $\mathbb{G}^3$ represented as 3×3 matrices or column vectors. The alternatives are best explained by a representative example.

With an obvious change of notation, we can write equation (103) for the change in *screw coordinates* induced by a shift $\mathbf{r} = \mathbf{x}_Q - \mathbf{x}_P$ from base point P to point Q in the form

$$V_Q = e\mathbf{v}_Q - i\boldsymbol{\omega} = \underline{T}_{\mathbf{r}} V_P = e(\mathbf{v}_P - \mathbf{r}\times\boldsymbol{\omega}) - i\boldsymbol{\omega}\,. \tag{122}$$

This has a matrix representation $\hat{V}_Q = \hat{\underline{T}}_{\mathbf{r}} \hat{V}_P$ with the explicit form

$$\begin{bmatrix} \mathbf{v}_Q \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} 1 & -\mathbf{r}\times \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_P \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_P - \mathbf{r}\times\boldsymbol{\omega} \\ \boldsymbol{\omega} \end{bmatrix}. \tag{123}$$

Similarly, we can write equation (100) for the induced change of *coscrew coordinates* in the form

$$W_Q = -e_0\mathbf{f} + i\boldsymbol{\Gamma}_Q = \underline{T}^*_{\mathbf{r}} W_P = -e_0\mathbf{f} + i(\boldsymbol{\Gamma}_P + \mathbf{r}\times\mathbf{f}) \tag{124}$$

Its matrix representation $\hat{W}_Q = \hat{\underline{T}}^*_{\mathbf{r}} \hat{W}_P$ has the explicit form

$$\begin{bmatrix} \mathbf{f} \\ \boldsymbol{\Gamma}_Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\mathbf{r}\times & 1 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\Gamma}_P \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\Gamma}_P + \mathbf{r}\times\mathbf{f} \end{bmatrix}. \tag{125}$$

These four equations suffice to show how any equation in the literature on screw theory or robotics can be translated into CGA and vice versa. For example, in [18] the screws in (123) and coscrews in (125) are represented as 6-component column vectors.

This example reveals a significant drawback of the matrix representations: the matrices do not encode the distinction between screws and coscrews, which, in contrast, is explicitly expressed by the distinction between e and e_0 in (122) and (124). Expressed in more general terms: the matrix representations suppress the geometric meaning of matrix elements, which is explicitly encoded in the algebra $\mathbb{G}^{1,1}$. Furthermore, the matrix representation (121) implicitly forces one to adopt a conformal split, which means, as we have seen, one is forced into a covariant rather than invariant approach to geometry. Nevertheless the isomorphism $\mathbb{G}^{1,1} \simeq \mathbb{M}_2(\mathbb{R})$ is essential for relating CGA to the robotics literature.

XIV. Linked Rigid Bodies & Robotics

The potential for application of CGA to robotics is best illustrated by a simple example. Fig, 14 depicts a kinematic chain with three segments in a *reference pose*:

$$x_0 = e_0 + a + b + c \tag{126}$$

Rotations at its three joints are specified by *twistors* $\{R_1, R_2, R_3\}$. Rotation at the first joint gives

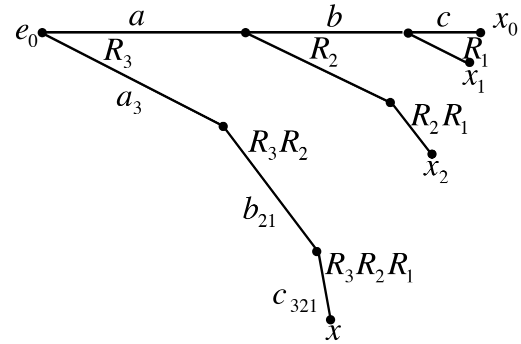

Fig. 14

$$x_1 = e_0 + a + b + R_1 c R_1^{-1} = e_0 + a + b + c_1 . \quad (127)$$

Subsequent or concurrent rotation at the second joint gives

$$x_2 = e_0 + a + R_2(b + R_1 c R_1^{-1})R_2^{-1} = e_0 + a + b_2 + c_{21} . \qquad (128)$$

Finally, the net result of rotations at all the joints is *general pose*:

$$x = e_0 + R_3[a + R_2(b + R_1 c R_1^{-1})R_2^{-1}]R_3^{-1} = e_0 + a_3 + b_{32} + c_{321} . \qquad (129)$$

A conformal split with the fixed point e_0 gives the position vector for the endpoint:

$$\mathbf{x} = x \wedge E = R_3[\mathbf{a} + R_2(\mathbf{b} + R_1 \mathbf{c} R_1^{-1})R_2^{-1}]R_3^{-1} = \mathbf{a}_3 + \mathbf{b}_{32} + \mathbf{c}_{321} . \qquad (130)$$

Of course, the rotations need not be confined to a plane (as presumed in Fig. 14 for simplicity of illustration). Restrictions on the range of motion at each joint are encoded in the twistors. For example, a rotation R_1 with one degree of freedom can be given the angular form

$$R_1 = \exp\left\{-\frac{1}{2}\mathbf{n}_1\alpha_1\right\} \qquad \text{where} \qquad 0 \le \alpha_1 \le \pi \qquad (131)$$

and unit vector $\mathbf{n}_1$ is the direction of the joint axis for a right handed rotation.

Kinematics of the chain can be described a follows. Irrespective of how the joints are characterized, the twistors satisfy equations of the form

$$\dot{R}_k = -\tfrac{1}{2} i \boldsymbol{\omega}_k R_k . \qquad (132)$$

Hence for $R_{32} = R_3 R_2$ we have

$$\dot{R}_{32} = -\tfrac{1}{2} i \boldsymbol{\omega}_{32} R_{32} \qquad \text{with} \qquad \boldsymbol{\omega}_{32} = \boldsymbol{\omega}_3 + R_3 \boldsymbol{\omega}_2 R_3^{-1} . \qquad (133)$$

Finally, with $\boldsymbol{\omega}_{321} = \boldsymbol{\omega}_3 + R_3 \boldsymbol{\omega}_2 R_3^{-1} + R_3 R_2 \boldsymbol{\omega}_1 R_2^{-1} R_3^{-1}$, for the derivative of the end point position vector (130), we get

$$\dot{\mathbf{x}} = \boldsymbol{\omega}_3 \times \mathbf{a}_3 + \boldsymbol{\omega}_{32} \times \mathbf{b}_{32} + \boldsymbol{\omega}_{321} \times \mathbf{c}_{321} \qquad (134)$$

This is only the beginning for application of CGA to robotics, but we have all the theoretical machinery we need for any task. To incorporate dynamics we introduce the inertia properties of each body with (95) and the applied wrenches with (97). Moreover, CGA offers a promising approach to modeling complex interactions between bodies, such as viscoelastic coupling at joints [2].

The next phase in the development of *CGA robotics* is detailed applications to specific problems. This development is already underway by attendees at this conference and others in the GA community. However, I see a need for more systematic mining of the robotics literature to incorporate established problems, results and methods in CGA and promote broader interaction within the engineering community. I thought about offering suggestions for literature to consult. But the robotics literature is so vast and variable in complexity and quality that I fear my suggestions could be as misleading as helpful. Consequently, I add only one recent reference [19] to those I have already mentioned.

To sum up, CGA provides a powerful mathematical framework for robotics R & D with the twin goals of (1) simplicity and clarity in mathematical formulation, (2) efficiency and speed in computation.

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