Mathematical Modeling ICTMA-13: Education and Design Sciences

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MODELING THEORY
for **Math** and **Science Education**

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Abstract: *Mathematics has been described as the science of patterns. Natural science can be characterized as the investigation of patterns in nature. Central to both domains is the notion of model as a unit of coherently structured knowledge. Modeling Theory is concerned with models as basic structures in cognition as well as scientific knowledge. It maintains a sharp distinction between mental models that people think with and conceptual models that are publicly shared. This supports a view that cognition in science, math, and everyday life is basically about making and using mental models. We review and extend elements of Modeling Theory as a foundation for R&D in math and science education.*

1. INTRODUCTION

Why should a theoretical physicist be concerned about mathematics education? My answer will be a long one, but let me begin by introducing you to some of my esteemed colleagues in Box 1. These fellows are such good physicists that most if not all of them would be worthy candidates for a Nobel Prize if they were alive today. You may know that they are quite good at mathematics as well! Indeed, mathematics textbooks often count them as mathematicians without mentioning that they are physicists. I dare say, however, that they would be mightily offended to hear that they are not counted as physicists. Likewise, I am more than miffed when reviewers of my math-science education proposals discount my qualifications as a mathematician because my doctorate is in physics. Like the fellows in the list, I regard my scientific research as equal parts mathematics and physics. The fact that the education establishment does not recognize that theoretical physicists are uniquely well-qualified to address education at the interface between mathematics and science is traceable to a serious problem within the mathematics profession itself.

Newton
Euler
Gauss
Lagrange
Laplace
Cauchy
Poincaré
Hilbert
Weyl
Von Neumann

Box 1: Some distinguished theoretical physicists

Of course, the list of *physicist/mathematicians* in Box 1 is far from complete. Many of my favorite colleagues are omitted. But there are at least two good reasons why the list ends in the middle of the twentieth century. The distinguished Russian mathematician V. I. Arnold put his finger on both in a widely circulated diatribe *On Teaching Mathematics* [1], wherein he asserts

> "Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap. "

> "In the middle of the 20th century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course in total ignorance of other sciences."

Mathematician-cum-historian Morris Kline has thoroughly documented "the disastrous divorce" of the mathematics profession from physics, which began in the latter part of the nineteenth century [2]. He estimated that, by 1980, eighty percent of active mathematicians were ignorant of science and perfectly happy to remain that way.

The divorce is thus an incontrovertible fact, but how disastrous can it be if only a minority of mathematicians like Arnold and Kline are alarmed? Isn't it a natural consequence of necessary specialization in an increasingly complex society? And isn't Arnold's claim of intimacy between math and physics merely a personal opinion? Surely the majority of mathematicians believe that mathematics is a completely autonomous discipline.

I claim that the answer to all these questions is a resounding NO! Indeed, *I submit that the single most serious deficiency in U.S. math education is the divorce of mathematics from physics* in the education of mathematicians, in the training of math teachers, in the structure of the K–12 (–16–20) curriculum. Moreover, this is not a simple deficiency in the breadth of education; it is a fundamental problem in conceptual learning and cognition. I claim that *cognitive processes for understanding math and physics are intimately linked and fundamentally the same*! Indeed, I claim that ***Physics is cognitively basic*** *to quantitative science in all domains!!*

Before delving into the deep cognitive issues, let us note some obvious academic consequences of the math/physics divorce. Training in mathematics is essential for all physicists, amounting to the equivalent of a dual major in mathematics for theoretical physicists. But math courses have become increasingly irrelevant to physics, so physics departments offer their own courses in "Methods of mathematical physics" at both graduate and undergraduate levels, with additional courses in more specialized topics like group theory. One consequence is a narrowing of the physicist's appreciation of mathematics. But a far more serious consequence is the reduction in opportunity for math majors to learn about vital connections to physics. This continues through graduate school, so the typical math PhD is ill-prepared for work in applied mathematics. Some math departments have attempted to remedy this deficiency with courses in mathematical modeling, but mathematical modeling without science is like the Cheshire cat: form without content! This is one of the deep cognitive issues that we need to address.

Far and away the most serious consequence of the math/physics divide is the deficient preparation of K-12 math teachers! The neglect of geometry and excess of formalism that Arnold [1] deplores in the university math curriculum has propagated to teacher preparation. There is abundant evidence that most teachers see their job as teaching formal rules and algorithms. Few have even a minimal understanding of Newtonian physics, so most are inept at applying algebra and calculus even to simple problems of motion. Consequently, high school physics courses are forced to revisit the prerequisite math knowledge that students are supposed to bring from years of math instruction. As the math courses lack the intuitive base necessary for conceptual understanding, students are forced to rote learning, which has a short half-life, so their recollection of math has decayed to nearly to zero by the time they get to college.

I doubt that these crippling deficiencies in math education can be fully resolved without a "sea change" in the culture of mathematics. To drive such revolutionary change we need a coherent theory of mathematical learning and cognition supported by a substantial body of empirical evidence. My purpose here is to report on progress in that direction.

2. ORIGINS OF MODELING THEORY

I have been investigating the epistemology of science and mathematics across the full range of academic disciplines for half a century. As that may sound implausible, let me describe the unusual initial conditions that got me started.

My father was an accomplished mathematician who helped organize American mathematicians to support the war effort in WWII. Consequently, he got to know mathematicians across the country on a first name basis. That served him well when, shortly after the war, he was wooed from the University of Chicago to build a first rank math department at UCLA. He was also appointed director of the Institute for Numerical Analysis (INA), where the National Bureau of Standards installed the first electronic computer in western United States. With a solid background in "pure mathematics" (Calculus of Variations), he blossomed then into a pioneer in the fledgling fields of Control Theory and Numerical Analysis, for which he was posthumously inducted into the Hall of Fame for Engineering, Science and Technology (HOFEST). The well-funded, vigorous research activity at the INA and the rapid emergence of the UCLA math department attracted a steady stream of distinguished mathematicians from around the world, for which my father was usually the host. He was at the acme of his career when I entered graduate school in 1956.

My undergraduate major in philosophy introduced me to the great conundrums of epistemology, and I was inspired by Bertrand Russell to switch to physics in search of answers. When I started graduate school, my father found me an office in the INA where I was surrounded by a whirlwind of excited activity about the beginnings of Computer Science and Artificial Intelligence. That prepared me to follow the evolution of both those fields throughout my career, though my main efforts were concentrated on physics and mathematics. By my third year I hit upon "Geometric Algebra" as the central theme of my scientific research. That induced me to reject Russell's logicist view of mathematics and sharpened my insight and interest in epistemological and cognitive foundations. Throughout my graduate years in physics I spent most of my time in the math department where I imbibed the culture of mathematics. This strong association with mathematics continued throughout my career in physics, anchored by relations with my father.

My diverse interests in cognitive science and theoretical physics converged in the 1980's when I got embroiled in problems of improving introductory physics instruction. Like any confirmed theoretician, I framed the problems in the context of developing a theory with testable empirical consequences. Largely from my own experience as a scientist I identified scientific models and modeling as the core of scientific knowledge and practice, and I proceeded to incorporate it into the design of physics instruction with the help of brilliant graduate students Ibrahim Halloun and Malcolm Wells. Thus began a program of educational R&D guided by an evolving research perspective that I called **Modeling Theory** in a 1987 paper. That program has continued to evolve beyond all my expectations. An up-to-date review of Modeling Theory is available in a recent paper [3]. The present paper is a continuation of [3], introducing new material with only enough duplication to make it reasonably self-contained. Therefore, it contains many gaps, some of which can be filled by consulting [3], and others that I hope will stimulate original research. For Modeling Theory is an enormous enterprise that amounts to a thematic approach to the whole of cognitive science. The best we can do here is sample the major themes.

As schematized in Figure 1, research on Modeling Theory has developed along two complementary strands. The strand on the right investigates *scientific models and modeling practices* that are explicit and observable. It provides *a window to structure and process in scientific and mathematical thinking* that we aim to peek through. That involves us with the strand on the left, which will be our main concern.

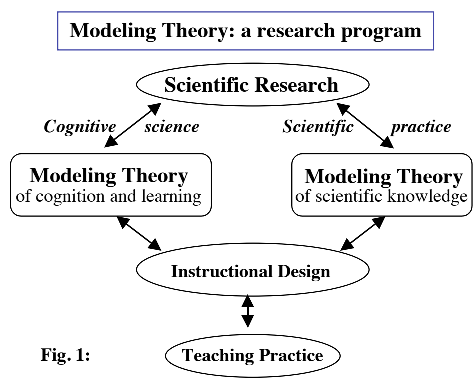

Fig. 1:

You may ask, *"Why should one adopt a model-centered epistemology of science?"* There are three good reasons:

1. <u>Theoretical</u>: Models are **basic units of coherently structured knowledge,** from which one can make *logical inferences, predictions, explanations, plans and designs.* One cannot make inferences from isolated facts or theoretical principles. A model can serve as inferential tool for the kind of structure it embodies.
2. <u>Empirical</u>: Models can be **directly compared with physical things and processes**. A theoretical hypothesis or general principle cannot be tested empirically except through incorporation in a model. Empirical data is

meaningless without interpretation supplied by a model.

3. Cognitive: Model structure is concretely **embodied in physical intuition,** where it serves as an element of *physical understanding.*

The third reason is based on the Modeling Theory of cognition set forth in this paper.

3. MODELS AND CONCEPTS

The term *model* is usually used informally (hence ambiguously), but to make crucial theoretical distinctions we need precise definitions. Although I have discussed this issue at length before [3], it is so important that I revisit it with a slightly different slant. I favor the following general definition:

A **model** is a **representation of structure** in a given **system**.

A *system* is a set of related *objects*, which may be real or imaginary, physical or mental, simple or composite. The *structure* of a system is a set of relations among its objects. The system itself is called the *referent* of the model.

We often identify the model with its *representation* in a concrete inscription of words, symbols or figures (such as graphs, diagrams or sketches). But it must not be forgotten that the inscription is supplemented by a system of (mostly tacit) rules and conventions for encoding model structure. As depicted in Figure 2, I use the term *symbolic form* for the triad of elements defining a model. I chose the term deliberately to suggest association with the great work on symbolic forms by philosopher Ernst Cassirer [4].

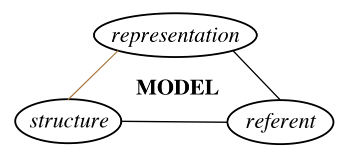

Fig. 2: Symbolic form of a model

We are especially interested in scientific models, for which I have often used the definition:

A (scientific) **model** is a **representation of structure** in a physical system or process.

This differs from the general definition only in emphasis and scope. Its scope is limited by assuming that the objects in a physical system are physical *things*. Nevertheless, the definition applies to all the sciences (including biology and social sciences). Models in the various sciences differ in the *kinds of structure* that they attribute to systems. The term *process* is included in the definition only for emphasis; it refers to a change in the structure of a system. Thus a *process model* is an abstraction of structural change from a more complete model including objects in the system.

In most discussions of scientific models the crucial role of structure is overlooked or addressed only incidentally. In Modeling Theory ***structure is central*** to the concept of model. The *structure* of a system (hence structure in its model) is defined as a *set of relations* among the objects in the system (hence among parts in the model).

Universal structure types:

From studying a wide variety of examples, I have concluded that **five types of structure suffice** to characterize any scientific model. As this seems to be an important empirical fact, a brief description of each type is in order here.

- **Systemic structure:** Its representation specifies (a) *composition* of the system (b) *links* among the parts (individual objects), (c) links to *external agents* (objects in the environment). A diagrammatic representation is usually best (with objects represented by nodes and links represented by connecting lines) because it provides a wholistic image of the entire structure. Examples: electric circuit diagrams, organization charts, family trees.

- **Geometric structure:** specifies (a) *configuration* (geometric relations among the parts), (b) *location* (position with respect to a reference frame)

- **Object structure:** *intrinsic properties* of the parts. For example, mass and charge if the objects are material things, or *roles* if the objects are *agents* with complex behaviors. The objects may themselves be systems (such as atoms composed of electrons and nuclei), but their internal structure is not represented in the model, though it may be reflected in the attributed properties.

- **Interaction structure:** properties of the links (typically *causal* interactions). Usually represented as binary relations on object pairs. Examples of interactions: forces (momentum exchange), transport of materials in any form, information exchange.

- **Temporal (event) structure:** *temporal change in the state* of the system. Change in position (motion) is the most fundamental kind of change, as it provides the basic measure of time. Measurement theory specifies how to quantify the properties of a system into property variables. The state of a system is a set of values for its property variables (at a given time). Temporal change can be represented *descriptively* (as in graphs), or *dynamically* (by equations of motion or conservation laws).

Optimal precision in definition and analysis of structure is supplied by **mathematics, the science of structure.** This agrees with the usual notion of a *mathematical model* as a representation in terms of mathematical symbols.

Now, here is a **perplexing question** that bothered me for decades: If the meaning of a scientific model derives from its physical interpretation, *from whence comes the meaning of a mathematical model*? Mathematical models are **abstract**, which means they have **no physical referent!**

It dawned on me during the last decade that the emerging field of **cognitive linguistics** provides a **revolutionary answer**. Cognitive linguistics has revolutionized the field of semantics by maintaining that the actual ***referents of language are mental***

models in the mind rather than concrete objects in an external world. It follows that, if mathematics is "the language of science," then the *referents of mathematical models must be mental models.* Likewise, *the proper referents of scientific models must be mental models of physical situations*, which are only indirectly related to real physical systems through data, observation and experiment.

This implies a common cognitive foundation for math, science and language: Just as science is about making and using objective models of real things and events, so cognition (in mathematics and science as well as everyday life) is about making and manipulating mental models of imaginary objects and events!

Let me sum up this revolution in semantics with a modified definition:

A **conceptual model** is a **representation of structure** in a **mental model.**

As before, the representation in a conceptual model is a concrete inscription that encodes structure in the referent. However, we make no commitment as to what the structure of a mental may represent. Henceforth, scientific and mathematical models are to be regarded as conceptual models. But the referent of a conceptual model is always a mental model, so its structure in the mind is inaccessible to direct observation. How, then, can this be an advance in Modeling Theory?

The answer is: It enables transfer from a Modeling Theory of scientific knowledge to a Modeling Theory of cognition in science and mathematics. Much is known about the structure of scientific models. We seek to solve the inverse problem of inferring the structure of mental models from the objective structure of scientific representations. If that seems like an impossible task, note that it is commonplace to infer thoughts in other minds from social interaction. Can we not make stronger inferences with the full resources of science? Here we have a modeling approach to the theory of cognition, so we can draw on the whole corpus of results in cognitive science for support and critique. I will not duplicate my previous reference to that enormous literature [3]. However, I should emphasize the special relevance of cognitive linguistics and point out that two recent introductions to the field [5, 6] provide a comprehensive overview that was difficult to put together only a few years ago.

Let me now propose the **First Principle for a Modeling Theory of Cognition:**

I. *Cognition is basically about making and manipulating mental models.*

I call this the **Primacy (of Models) Principle**, noting I have already tacitly invoked a variant of it for the Modeling Theory of scientific knowledge. Commitment to this principle might seem extreme, for I must admit that it is not to be found in the cognitive science literature from which I draw most of the supporting evidence. However, I contend that for a guiding research principle the standard is not that it is true but that it is *productive*, by which I mean that it leads to significant predictions that are empirically testable. Even if proved wrong, that would be quite an interesting result! In the meantime, we shall see that the primacy principle can carry us a long way.

For a start, the Primacy Principle helps sharpen the definition of a concept, as it implies that concepts must refer to mental models, at least indirectly. As done before [3], I define a ***concept*** as a **{form, meaning} pair** represented by a **symbol** (or assembly of symbols). In analogy to Fig. 2, I define the ***symbolic form*** of a concept as the triad in Fig. 3. Much like a model, the *form* of a concept is its conceptual structure, including relations among its parts and its place within a conceptual system. The *meaning* of a concept is its relation to mental models. All this is close enough to the usual loose definition of "concept" to conform to common parlance. It provides then a foundation for a more rigorous analysis of important concepts.

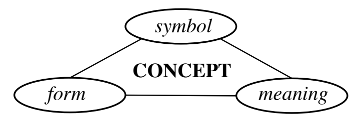

Fig. 3: Symbolic form of a concept

We are now prepared to propose the **Second Principle for a Modeling Theory of Cognition:**

II. *Mental models possess five basic types of structure: systemic, geometric, descriptive, interactive, temporal.*

I call this the **Principle of Universal Forms,** where the forms are the five types of structure. Obviously, this is direct transfer to mental models of the structural types identified above for scientific models. Thus, it provides us immediately with a rich system of conjectures about mental models to investigate and amend if necessary. Moreover, it brings along a rich system of basic concepts involved in characterizing the forms.

Scholars will note strong similarity of the Universal Forms to Immanuel Kant's "Forms of Intuition" and "Pure Concepts of the Understanding (Categories)" [7]. This should not be surprising, since Kant engaged in a similar analysis of cognition with special attention to the mathematics and physics of his day. Kant proposes his Categories as a complete list of universal forms for logical inference. In Modeling Theory this should translate into universal forms for the *synthesis* (to use Kant's term) of mental models. A brief account of Kant's "transcendental" knowledge analysis is given below, but detailed comparison with Modeling Theory will not be attempted here. Today we have so much more factual information about the structure of science and cognition to guide and support our conjectures. Even so, the relevance of Kant's thinking to current cognitive science has been examined by Lakoff and Johnson [8].

The Universal Forms are also similar to semantic structures identified in cognitive linguistics, especially in the work of Leonard Talmy [5]. This is another rich area for comparative research that cannot be pursued here, though we shall touch on more ideas to throw into the mix.

Modeling Theory must ultimately account for the origins of structure in mental models and its elevation through the creation of symbolic forms into shareable concepts and conceptual systems. Let me comment on the second part of this ambitious research agenda. Taking for granted the existence of structured mental models in perceptual experience, we posit the human ability to make distinctions with respect to similarities and differences in model structure as the basic mechanism for creating *category concepts*.

Cognitive research has established that there are two general types of category concepts, which I shall distinguish by the non-standard terms *implicit* and *explicit* to emphasize an important point. *Implicit concepts* are determined by their mental referents, that is, they *derive meaning* from a web of associations with one or more mental models. For example, the concept *dog* derives meaning from a stored mental image of a prototypical dog. Most category concepts in natural language are of this type [5], though my brief comments do not do justice to the subject. Implicit concepts could well be called *empirical concepts*, because their structures are built from experience in the mind of each individual.

In contrast to implicit concepts, which are grounded in private mental images, *explicit concepts are defined* by public representations. For explicit concepts, category membership is *defined by a set of necessary and sufficient conditions*. This is, of course, the classical concept of category that we inherited from Aristotle. It was only recently realized that ordinary (i.e. implicit) concepts are not of this type. Nevertheless, the crucial concepts of science and mathematics must be of the explicit type to qualify as objective knowledge.

4. IMAGINATION AND INTUITION

Modeling Theory provides a foundation for precise definitions of important concepts in cognitive psychology. Human imagination is one such concept, important and familiar to everyone, but elusive in cognitive science. Let us reinvigorate it here with a definition that embodies the First Principle of Modeling Theory:

Imagination is *the faculty for making and manipulating mental models.*

This squares well with a well-established line of research on *narrative and discourse comprehension*, which supports the view that the *linguistic function of words is to* activate, elaborate and modify mental models of objects and events in an imaginary unfolding scene [9]. We are most interested here in the thesis that the very same cognitive process is involved in thinking mathematics and physics. To sustain that thesis we must account for the unique features of cognition in the scientific domain.

Since the latter part of the nineteenth century, mathematicians and philosophers have vigorously debated the foundations of mathematics with no sign of consensus [10. 2]. But all agree on a crucial role for *mathematical intuition*. Even the supreme formalist, David Hilbert, approvingly quoted Kant's famous aphorism: "All human knowledge begins with intuitions, thence passes to concepts and ends with ideas." Though mathematical intuition is never mentioned in formal publications, it often comes up in informal discussion among mathematicians, and subtle hints of its presence appear in choices of mathematical terms and symbols. Recently, however, Lakoff and Núñez [11] have dared to shine the light of cognitive science on the recesses of mathematical thought. My aim is to do the same from the perspective of Modeling Theory.

Physical intuition is privately held in the same high regard by physicists that mathematicians attribute to *mathematical intuition.* I submit these two kinds intuition are merely two different ways to relate products of imagination to the external world, as indicated in Fig. 4.

Physical intuition matches structure in mental models with structure in physical systems. **Mathematical intuition** matches mental structure with symbolic structure. Thus, structure in the imagination is common ground for both physical and mathematical intuition.

I surmise that physical intuition is highly developed among experimental physicists, involving detailed mental images of experimental design, equipment, measurement procedures and data analysis. None of that is involved in mathematical intuition, but theoretical physics requires integrating a good deal of both. Supporting evidence for this point will emerge as we move on.

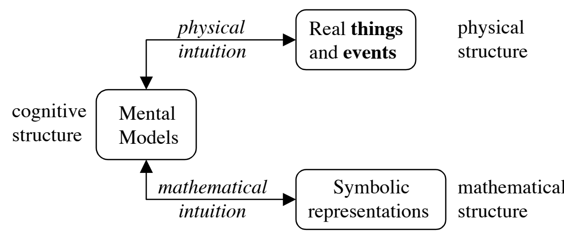

Fig. 4: Intuition of structure

Identification of **intuition** as the bridge between imagination and perception is a secure starting place for exploring the specifics of mathematical intuition. It would be helpful to have the testimony of proficient mathematicians to guide the exploration. Much anecdotal testimony is scattered throughout the literature, but it would take a major act of scholarship to bring it together. We shall have to be satisfied with a few telling examples.

Mathematician Jacques Hadamard [12] surveyed 100 leading physicists (about the year 1900) and gives an introspective account of his own thinking, as well as that of others including Poincaré and Einstein. He documents two major facts about mathematical thinking: at the conscious level, much of it is imagistic without words; and, much of it is done unconsciously, with clear insights or solutions emerging with "sudden spontaneousness" into conscious thought. He does not discriminate between the thinking of mathematicians and physicists. He quotes from a letter by Einstein:

> "The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. . . . The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be voluntarily reproduced and combined. . . . The above-mentioned elements are, in my case, visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary state.. . ."

It is noteworthy that Einstein is famous for inventing thought experiments that proposed new relations between theory and experiment. When this is presented as evidence for his singular genius, it remains unremarked that invention of a thought experiment is an essential early step in the design of any experiment.

All this suggests that "free play of the imagination"(as Einstein put it) has a far more significant role in math/science thinking (and human reasoning in general) than is commonly recognized in educational circles. Most intuitive structure is represented subliminally in the cognitive unconscious and is often manifested in pattern recognition and conceptual construction skills. Finally, it should be noted that Einstein's description supports our view that intuition is grounded in the sensory-motor system; moreover, that ideas may be generated in the imagination before they are elevated to concepts by encoding in symbols.

Hadamard's report provides empirical support for general features of mathematical imagination and intuition, but it lacks the detail we need to describe its structure. To remedy that, we can do no better than turn to Kant's "transcendental analysis" in the *Critique of Pure Reason* [7]. He was not a professional mathematician, but he did teach mathematics and physics for fifteen years before writing the Critique. Moreover, his analysis was greatly respected and highly influential among mathematicians throughout the nineteenth century and beyond. My attempt to present the nub of Kant's argument is indebted to clarifications by philosopher Quassim Cassam [13].

Kant conceded to his empiricist predecessors Locke, Hume and Berkeley that all knowledge is derived from experience, but he rejected attempts to derive certain knowledge from that. Rather, he turned the problem of knowledge on its head and accepted Euclid's geometry and Newton's physics as objective facts. He then asked the trenchant question "How is such knowledge possible?" This posing of a "how-possible question" (as Cassam calls it) is the essential first step in Kant's "*transcendental*" approach to epistemology. He completes his argument with a multi-level answer.

Kant applies his transcendental method to a number of epistemological problems. But the test case is Euclidean geometry, as that was universally acknowledged as knowledge of the most certain kind. Thus, he asked: "How is geometrical knowledge possible?" This is just the kind of question we want to answer in detail. Kant begins his answer by identifying ***construction in intuition*** as ***a means*** for acquiring such knowledge:

> "Thus we think of a triangle as an **object**, in that we are conscious of the combination of the straight lines according to a **rule** by which such an **intuition** can always be **represented**. . . This representation of a universal procedure of imagination in providing an image for a concept, I entitle the **schema of this concept**."

Kant did not stop there. Like any good scientist he anticipated objections to his hypothesis. Specifically, he noted that his intuitive image of a triangle is always a *particular triangle*. How, he asks, can construction of a concept by means of a single figure "express universal validity for all possible intuitions which fall under the same concept?" This is the general epistemological *problem of universality* for the case of Kant's theory of geometrical proof. Kant's notion of geometrical proof is by construction of figures, and he argues that such proofs have universal validity as long as the figures are "determined by certain universal conditions of construction." In other words, construction in intuition is a ***rule-governed activity*** that makes it possible for geometry to discern "the universal in the particular."

Kant wants more. What still needs to be explained is the *capacity* of pure intuition to provide geometrical knowledge. Kant's argument leads ultimately to the conclusion that space itself is an "*a priori* intuition" that "has its seat in the subject only." He concludes famously that *space and time* are "*a priori forms of intuition*," intrinsic features of mind that shape all experience.

We need not follow Kant's argument to conclusions that have since proven to be untenable, such as the claim that geometry of the physical world must be Euclidean because our minds cannot conceive otherwise. We now know that there are many kinds of non-Euclidean geometry, and the geometric structure of space-time is an empirical matter to be settled by interplay between theory and experiment. The bottom line is that Kant's hypothesis of spatio-temporal constraints on cognition is still viable today, but it must be recognized as an empirical issue to be settled by research in cognitive science.

There is much more to be said in favor of Kant's analysis. First, his characterization of geometric intuition has been universally approved (or, at least, never challenged) by mathematicians even to present day, as it is easily adapted to any non-Euclidean geometry by simple changes in the rules. Second, his argument that **inference from the particular to the universal is governed by subsumption under rules** is a profound insight that has not attracted the attention it deserves, even, it seems, from devoted Kantian scholars. Its import is evident in Modeling Theory, for it determines a mapping of structure in mental constructions (models) to structure in drawn figures, propositions or equations. That is, rules for parsing and manipulating mental constructions correspond directly to rules for constructing and manipulating mathematical representations. This is evidently *a basic mechanism in mathematical intuition* as I defined it earlier. Moreover, it is *a means for constructing and sharing objective knowledge*, as the rules are publicly available to everyone, though it is nontrivial to learn how to employ them.

The power of rules was so evident to Kant that he posited a ***faculty of judgment*** to administer it: "If understanding as such is explicated as our power of rules, then the power of judgment is the ability to subsume under rules, i.e., to distinguish whether something does or does not fall under a given rule." Judgment developed into the central theme of Kant's philosophy, but in the abundance of its applications to morals, religion and aesthetics, its fundamental role in mathematical intuition and objective knowledge seems to have been lost.

We are now prepared for a more incisive comparison of mathematical and physical intuition. To begin with, the intuitive structure of Euclidean geometry is common knowledge for mathematicians and physicists. I submit, however, that their intuitions of geometry gradually diverge as they employ geometry in different ways. The mathematician concentrates on construction and analysis of formal structures. The physicist uses geometry for modeling rigid bodies and measurement of length, which is the foundation for physical measurements of every kind. Such developments in mathematics and physics do not have to go far before their common ground in Euclidean geometry is no longer obvious. With respect to the Kantian category of *causality*, intuitions of physicists and mathematicians diverge even more strongly, as we shall see.

It may be objected that our how-possible analysis of geometry is too limited for general conclusions about mathematical intuition. As a remedy, I recommend a how-possible analysis of set theory, group theory, algebra and any other mathematical system that the reader regards as fundamental. In fact, I submit that it would not be difficult and perhaps enlightening to frame the math concept analysis of Lakoff and Núñez [11] in how-possible terms.

5. MATHEMATICAL VERSUS PHYSICAL INTUITION

Let me reinforce our conclusions about mathematical intuition with testimony by Hilbert from an address delivered in 1927:

> "No more than any other science can mathematics be founded on logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from on another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else, nor requires reduction. This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. This is the very least that must be presupposed, no scientific thinker can dispense with it, and therefore everyone must maintain it consciously or not."

Note the coupling between concrete signs and intuitions, with logical inference grounded on the intuitive side.

For comparison, let's hear testimony about physical intuition from an eminent physicist. Heinrich Hertz (1857-1894) discovered the means to generate and detect electromagnetic radiation, surely one of the greatest experimental achievements of all time. He was equally accomplished as a theoretical physicist, though his tragic early death deprived the world of his genius. His profound grasp of cognitive processes in science is exhibited in the following passage [14]:

> "The most direct, and in a sense the most important problem which our conscious knowledge of nature should enable us to solve is the anticipation of future events, so that we may arrange our present affairs in accordance with such anticipation.
>
> . . "We form for ourselves images or symbols of external objects; and the form which we give them is such that the necessary consequents of the images in thought are always the necessary consequents in nature of the things pictured [Predictability]. In order that this requirement may be satisfied, there must be a certain conformity between nature and our thought.
>
> . . . " The images we form of things are not determined without ambiguity by the requirement of [Predictability].

[I have inserted the term [Predictability] to compress the link between his last two paragraphs.]

> Hertz goes on to explain that images are constrained by certain *Conformability Conditions,* including
>
> *Admissibility*: Images must not contradict the laws of thought.
> *Distinctiveness*: Images should maximize essential relations of the thing.
> *Simplicity*: Images should mimimize superfluous or empty relations.
> He adds that "Empty relations cannot be altogether avoided."
>
> Hertz then explains that *scientific representations* (of our images) satisfy different postulates.

This passage (condensed for brevity) is studded with brilliant insights. First, note that it is consistent with Kant's account of geometric intuition (with which Hertz was surely familiar), but it surpasses Kant in original detail. Next, note how sharply Hertz distinguishes between images (mental models) and their scientific representations. He emphasizes that to have predictive value the images must satisfy certain rules, which he sharply distinguishes from rules governing their public representations. Finally, note the implication from Hertz's first paragraph that the faculty of intuition has evolved to guide effective action in the environment. This is currently a major theme in the emerging field of evolutionary psychology.

Differences between mathematical and physical intuitions emerge in meanings attributed to mathematical expressions. We often speak of mathematical symbols as though they have unique meanings that are the same for everyone. But we know that meanings are private constructions in the imagination of each individual, so tuning them to agree among individuals is a subtle social process. We have noted that public access to geometrical figures provides common ground for geometric intuitions of both mathematicians and physicists. Now let us consider an important concept where intuitions strongly diverge, namely, the concept of force.

Modeling Theory

The general concept of *interaction* has been identified as one of the universal forms of knowledge in Modeling Theory. It corresponds closely to Kant's *causality* category. Though that category is construed broadly enough to include human volition, there is no doubt that the *Newtonian concept of force* was centermost in Kant's thinking. *Force dynamics* also appears as a major category in cognitive linguistics, especially in the work of Talmy and Langacker [5]. However, as I have explained before [3], linguistic research on the force concept has yet to be reconciled with physics education research (PER).

Divergence of student intuitions about force from the Newtonian (i.e. scientific) force concept are reliably measured by the *Force Concept Inventory* (FCI). FCI assessment on large populations of students from middle school to graduate school shows conclusively that before physics instruction student concepts diverge from Newtonian concepts in almost every dimension [3]. Moreover, most students are far from Newtonian even after a year of university physics. I surmise from this that mathematics professors who have neglected physics in their education will likewise retain naïve concepts of force. To check that out, it would be interesting to test a representative sample of such subjects with the FCI. But who dares bell the cat?

Naïve concepts of force have often been dismissed as misconceptions to be replaced by the scientific Newtonian concept. But that is a serious mistake stemming from a naïve view of cognition and learning. It should be recognized instead that student intuitions are essential cognitive resources developed through years of real world experience. We understand the world by mapping events into the mental spaces of our imagination. The chief problem in learning physics is not to replace intuitions but to ***tune*** the mapping to produce a veridical image of the world in the imagination.

To thoroughly understand what learning the Newtonian force concept entails, we need an inventory of intuitive resources that students bring to the experience. Andrea diSessa has pioneered identification and classification of basic intuitions of physical mechanism that he calls *phenomenological primitives*, or *p-prims* [15].

Without going into detail that is readily accessible in the literature, I wish to explain how diSessa's theory of p-prims (or, at least, something very much like it) fits naturally into Modeling Theory. It will be sufficient to comment on the p-prims listed in Fig. 5.

Force and Agency	**Constraint Phenomena**
OHM'S P-PRIM	*BLOCKING*
SPONTANEOUS RESISTANCE	*SUPPORTING*
FORCE AS MOVER	*GUIDING*
DYING AWAY	
Balance and Equilibrium	
DYNAMIC BALANCE	
ABSTRACT BALANCE	

Fig. 5: Force p-prims (from Sherin 2006)

Much like the image schemas in cognitive linguistics [5], p-prims are stable units of mental structure employed in the construction of mental models. Though diSessa identified the p-prims largely from interviews of scientifically naïve students, I agree with Bruce Sherin [16, 17] that the same *p-prims are involved in structuring the physical* *intuition of mature physicists*. I regard this conclusion as a *major milestone in cognitive science*, so I will return to it after discussing the intuitive foundations.

diSessa found that, for naïve students, each p-prim is a simple, separate and distinct knowledge piece called forth for explanatory purposes by situational cues. Collectively, the p-prims compose a loose conceptual system that diSessa described as "knowledge in pieces." In contrast, Newtonian force is a complex, multidimensional concept [3]. Let's consider how the p-prims can be integrated into an intuitive base for the Newtonian concept.

Many p-prims in Fig. 5 have familiar names. This should not be surprising, because they derive from common human experiences. However, diSessa has given the peculiar name *Ohm's p-prim* to the most important one in the lot. That does suggest a historical role in the creation of Ohm's law for electrical resistance. But its most basic role is in the intuition of force. Ohm's p-prim is schematized as an *agent working against a resistance to produce a result*. No doubt it originates in personal experience of pushing material objects, and it is projected metaphorically to other situations [3]. diSessa notes that it serves as general intuitive schema for qualitative proportional reasoning (hence its applicability to Ohm's law by metaphorical projection). diSessa also suggests that it provides intuitive structure for the physicist's understanding of $\mathbf{F} = m\mathbf{a}$, where the result of applying a force is acceleration (but not velocity, as is common in naïve conceptions). Note the considerable adjustment in intuition required for a veridical match of mental model with physical events (in accordance with Hertz's conformability conditions). Indeed, ability to discriminate between velocity and acceleration (qualitatively as well as quantitatively) is already a major advance over naïve thinking. The upshot for a physicist is: the equation $\mathbf{F} = m\mathbf{a}$ serves as a symbolic form for reasoning about force and acceleration.

The *force as mover* p-prim holds that the response to an applied force is motion in the direction of the force, but after the vectorial concept of acceleration has been mastered, intuition can be adjusted to associate that direction with acceleration instead of velocity. It then becomes integrated into the intuition of $\mathbf{F} = m\mathbf{a}$ for the physicist.

There is an implicit principle in the Newtonian conceptual system that could be called *Universality of Force*. This principle, which holds that *motion is influenced by forces only*, is never mentioned in physics textbooks, so it is not surprising that many students who have completed a standard physics course have failed to reconcile all their p-prims (Fig. 5) with the Newtonian force concept. Initially, naïve students do not recognize the familiar motion effects of *resistance* and *dying away* as due to forces. Likewise, they do not associate effects known as *blocking, supporting* and *guiding* with forces. The physicist's intuition retains the p-prims for all these effects while integrating them into a universal force concept that associates forces with all these effects.

Reasonable as this account of the relation between p-prims and physics intuition may seem (to me, at least), we need more detail and stronger evidence to support it. Happily, Bruce Sherin has produced an impressive corpus of supporting evidence in a landmark study [17, 16] on the role of *intuitive knowledge in quantitative* *problems with physics equations*. Sherin's study is noteworthy not only for its originality and results, but for the quality of his data and data analysis. It provides an exemplar for a productive line of further research. Sherin's data come from videotaping pairs of moderately advanced students collaboratively solving physics problems of moderate difficulty. Consequently, it documents some behaviors characteristic of expertise along with revealing examples of ongoing learning. Readers are referred to the original articles for details. I will comment on Sherin's findings with a twist to fit them into Modeling Theory.

Sherin's first finding was supporting evidence for diSessa's conjecture that p-prims come to serve as heuristic cues for formal knowledge as expertise develops. Of course, we must distinguish situational (or verbal) cues for p-prims from their intuitive structure. The observable cues may be retained with little change, while the p-prim structure must be adapted to the expert's intuition. Modeling Theory enables a stronger inference, namely, that p-prim structure must be integrated consistently into the structure of the expert's mental model for a given physical situation. Actually, the p-prim may first cue construction of a mental model, which is in turn coordinated with construction of a formal representation for the model.

Sherin also found evidence that p-prims can drive problem solving in a fairly direct means. He looked for tuning of p-prims (refining the sense-of-mechanism) during equation construction and use. He suggests that happens when competing p-prims are cued to activation during analysis of a physical situation. Compromise then occurs when both p-prims are seen to have validity. In modeling terms this can be described as tuning the p-prims to fit a coherent mental model.

Sherin's most important finding came from observing students construct equations from ideas of what they want the equations to express. They were appealing to common sense, not just following formal rules. For example, in problems involving equilibrium of forces Sherin noted activation of *balancing p-prims* (Fig. 5) and strong linking of these p-prims to equations. This promoted qualitative reasoning from terms in an equation without actually solving the equation. Sherin concluded that the students were using the terms to represent p-prims; they were using equations to represent and coordinate common sense knowledge. He crystallized this brilliant insight by inventing the notion of a ***symbolic form***, which he defined as a *symbol template* associated with a *conceptual schema* that specifies a few entities and relationships among them. Each template belongs to a symbolic system for representing the conceptual schema in an arrangement of symbols. His coding scheme for a catalog of symbolic forms is shown in Fig. 6.

I was astounded when I first heard Sherin talk about his symbolic forms recently, for his term *symbolic form* is not only identical to the one I introduced to clarify the definition of *concept*, but its meaning and purpose strikes me as essentially the same as mine in the present context. The equivalence of terms is evidently a coincidence, but the equivalence of concepts bespeaks a convergence of independent lines of research.

Consequently, it is a simple matter to merge Sherin's results into Modeling Theory, though I have not asked his permission.

The *Balancing Form* in Fig. 6 expresses equivalent effects of two competing influences, which is the essential structure of the *balancing p-prim*. The Form is the same whether the influences are forces or torques or whatever intuition suggests. The notion of *balancing an equation* probably originated from this p-prim. Though some mathematicians may dismiss that notion as a mathematically irrelevant metaphor, others hold that it is an indispensable intuitive foundation for understanding mathematical equality. This is an elementary instance of the tension between formalism and intuition in mathematical understanding.

Competing Terms Cluster		**Terms are Amounts Cluster**	
COMPETING TERMS	$\square \pm \square \pm \square \ldots$	*PARTS-OF-A-WHOLE*	$[\square + \square + \square \ldots]$
OPPOSITION	$\square - \square$	*BASE ± CHANGE*	$[\square \pm \Delta]$
BALANCING	$\square = \square$	*WHOLE - PART*	$[\square - \square]$
CANCELING	$0 = \square - \square$	*SAME AMOUNT*	$\square = \square$
Dependence Cluster		**Coefficient Cluster**	
DEPENDENCE	$[\ldots x \ldots]$	*COEFFICIENT*	$[x \square]$
NO DEPENDENCE	$[\ldots]$	*SCALING*	$[n \square]$
SOLE DEPENDENCE	$[\ldots x \ldots]$	**Other**	
Multiplication Cluster		*IDENTITY*	$x = \ldots$
INTENSIVE•EXTENSIVE	x × y	*DYING AWAY*	$\left[e^{-x\ldots}\right]$
EXTENSIVE•EXTENSIVE	x × y		
Proportionality Cluster			
PROP+	$\left[\frac{\ldots x \ldots}{\ldots}\right]$	*RATIO*	$\left[\frac{x}{y}\right]$
PROP-	$\left[\frac{\ldots}{\ldots x \ldots}\right]$	*CANCELING(B)*	$\left[\frac{\ldots x \ldots}{\ldots x \ldots}\right]$

Fig 6: Symbolic forms by cluster (from Sherin 2001)

The *Proportionality Forms* in Fig. 6 are especially important, as proportional reasoning is a critical skill in quantitative science, but it has remained distressingly difficult to teach. Sherin proposes separate forms $prop^+$ and *prop* for direct and inverse proportionality, and he hypothesizes that they are strongly connected with intuitions of *effort* and *resistance* through Ohm's p-prim. This is quite interesting for many reasons. Math educators and psychologists have explored a number of ways to develop intuitive understanding of proportional relations. Historically, the first robust understanding was grounded in geometric intuition of similar triangles. Archimedes was probably the first to understand the proportionality of torques in balancing. Gradually the analogy "*a* is to *b* as *c* is to *d*" was transmuted into the proportionality symbolic form $a/b = c/d$. All of these facts are well known and influential in math education. I mention these facts because *Ohm's p-prim is not included,* so it is not among the most likely influences on prior understanding of proportional reasoning by the students that Sherin observed. Thus, Sherin has introduced new insight into cognition of proportional relations.

To reconcile the diverse intuitions of proportional relations, I suggest that Sherin's Proportionality Forms have other referents (intuitive meaning) besides Ohm's p-prim, including those in the historical list I mentioned. Multiple meanings for words are common in natural language; that is known as *polysemy* in cognitive linguistics, which has elevated it to fundamental status in linguistic theory [5]. Accordingly, I submit that a physicist has a repertoire of many meanings that can be assigned to a mathematical form, depending on activation by situational cues. Representation of structure in diverse situations by a single mathematical form is no doubt a primary source of the great power in mathematical modeling, so instruction should be designed to cultivate it.

Most proposals for teaching proportional reasoning emphasize laying the intuitive foundation first, but Daniel Schwartz and colleagues [18, 19] argue for the reverse: using mathematical representation to refine intuition. Of course, it is fundamental in Modeling Theory that the mapping of structure between mental models and their mathematical representations goes both ways. Undoubtedly, mathematics plays a role in tuning intuition.

As a final point, Sherin suggests that "washing out of physical meaning is a fundamental feature of the move from intuitive physics to more expert knowledge." He notes what he calls a "fundamental tension" "between the homogenizing influence of algebra and the nuance inherent in intuitive physics." Though I don't subscribe to his "washing-out" metaphor, I submit that Sherin has observed a significant effect, namely, a special case of the *fundamental tension between abstract (mathematical) form and physical intuition* (or *physical interpretation*, if you will)! Let's call it the **Form-Content Tension** for future reference. I submit that this tension is basically about matching structure in mathematical models to observable structure in the world. It can be construed as tension between mathematical and physical intuition, as defined in connection with Fig. 4.

In understanding a physical equation, the physicist is always concerned with correlating the mathematical structure in the equation with intuitive structure in a mental model. This is known as ***interpreting*** the equation. The correspondence is a two-way mapping. In constructing an equation the physicist incorporates structure from a mental model of a physical situation (as Sherin observed students doing). Conversely, presuming that a given equation applies to a given physical situation, the physicist uses structure in the equation to structure a mental model of the situation. Let's call this ***reading*** the equation.

Sherin asserts that qualitative reasoning with equations is the hallmark of physics expertise! Perhaps so, but, as physicist Robert Romer [20] emphasizes, reading physics equations for understanding is a prerequisite. If mathematics is the language of physics, then reading the equations of physics must be much like comprehending a narrative, namely, constructing meaning in a mental model. I submit that all qualitative reasoning is based on mental models, with terms in equations serving as cues for structure in the models. Reasoning from a mental model is necessarily qualitative. No model, no reasoning!

To be sure, equations also serve a quantitative role unlike statements in natural language. Semi-quantitative estimation and dimensional analysis are essential skills for matching models with data, much valued by physicists! However, overemphasis on the quantitative encourages students to look at equations purely algorithmically. Consequently, students often come to a first course in physics with something like a *vending machine model of algebraic equations*, wherein the variables are slots for inserting numbers and the equal sign spits out the answer. For them, equations have no more meaning than the nonsense phrase "Twas brilig and the slithy toves." For them, reading an equation is no more than reciting words. How can we engage students in making sense of equations?

6. MODELING INSTRUCTION

Let me quote myself on the objectives of science instruction:

> ***The great game of science is modeling the real world***, and ***each scientific theory lays down a system of rules for playing the game.*** The object of the game is to construct valid models of real objects and processes. Such models comprise the content core of scientific knowledge. To understand science is to know how scientific models are constructed and validated. ***The main objective of science instruction should therefore be to teach the modeling game***. [21]

Modeling Theory has been developed with that purpose expressly in mind. Its implications for the design of curriculum and instruction have been thoroughly discussed in the literature reviewed in [3]. Some of the highlights are reviewed here to make connection with the present paper, which has addressed only structural aspects of scientific knowledge. Modeling Theory is equally concerned with procedural aspects of scientific knowledge, which it characterizes in terms of making and using scientific models.

<u>Implications for Curriculum Design</u>:

- The ***curriculum should be organized around models***, not topics! because models are ***basic units of coherently structured knowledge***, from which one can make direct inferences about physical systems and comparisons with experimental data.
- Students should become familiar with ***a small set of basic model as the content core*** for each branch of science, along with selected extensions to more *complex models*.

- Theory should be introduced as a system of general principles for constructing models with a specified domain of validity.

Implications for Instructional Design:

Students should learn ***a modeling approach to scientific inquiry***, including

- proficiency with conceptual modeling tools
- qualitative reasoning with model representations
- procedures for quantitative measurement
- comparing models to data.

Implementation and Evaluation of Modeling Instruction:

The above modeling principles for curriculum and instruction design have been fully implemented in a High School physics course, and intensive (3-week) summer workshops have been developed to train in-service teachers in the innovative techniques of *Modeling Instruction*. A series of such *Modeling Workshops* was continuously supported by the National Science Foundation for fifteen years, with unprecedented success on many measures, including student gains on the FCI, external evaluations, teacher satisfaction and buy-in. Although the Workshops are very demanding, their popularity is so great that more than 2,000 teachers have attended at least one; this is nearly 10% of all physics teachers in the U.S. Full details about the program and its evaluation are available in documents at the Modeling Instruction website [3].

A few more words about Modeling Instruction are needed to appreciate the unique features most responsible for its success. The ***Modeling Method*** of instruction is a student-centered inquiry approach guided by the teacher, as recommended by the National Science Education Standards. The big difference is that all stages of inquiry are *structured by modeling principles.* Typical inquiry activities (or investigations) are organized into ***modeling cycles*** about two weeks long.

Each investigation focuses on understanding a concrete physical system/process; for example, an oscillating block suspended by a spring. After class discussion to set the stage for the investigation, students are divided into research teams of three or four to design and carry out experiments, analyze results and prepare a report. The teacher subtly guides the entire inquiry process with questions, suggestions and challenges, introducing equipment, standard terms, conventions, and representational tools as needed. The students soon learn that the objective of the investigation is to formulate and evaluate a well-defined scientific model of the system in question. By the third time through a modeling cycle, the students have assimilated the procedural knowledge in modeling inquiry, and they proceed systematically in further investigations without help from the teacher. This leaves the teacher free to concentrate on guiding the students to a clear understanding of the conceptual structure in the models they develop. The primary guidance mechanism is **modeling discourse:** which means that the teacher promotes *framing all classroom discourse in terms of models and modeling.* The aim is to sensitize students to the structure of scientific knowledge, in both declarative and procedural aspects. Of course, *the skill and understanding of the teacher* are the main factors in success of the Modeling Method. Consequently, the Modeling Workshops are designed to promote and curriculum materials have been developed to support it.

Design of the modeling cycle needs to be described in more detail to see how modeling structure is incorporated. For instructional purposes, modeling inquiry can be decomposed into four major phases: model *construction, analysis, validation*, and *deployment*. Each phase deserves separate commentary. But it should be understood that emphasis on various phases in the cycle may vary greatly, depending on objectives of the inquiry. Moreover, the phases are not necessarily implemented in a linear order; for example, questions raised in the analysis or validation phase may lead to modifications in the construction phase.

Model construction (or *development*) incorporates into the design of a conceptual model some or all the universal forms delineated in Section 2. Students and even teachers are not informed of this fact. Rather, they are introduced to *representational tools* and engaged in using the tools to model structure in concrete systems. Of course, that is not unique to Modeling Instruction. Using the tools of analytic geometry to model geometric structure and differential equations to model temporal structure is commonplace and often indispensable. However, Modeling Theory coordinates application of all the various tools toward construction of a complete and coherent scientific model of any real situation. This has led to significant improvements in the conceptual process of model construction. For example, recognition that *specification of systemic structure is an essential first step* in constructing any model. That step consists of first identifying the composition and interactions of the system to be modeled, and second creating a diagram (which I call a *system schema*) to represent that information. The second part of that step is often overlooked, with the consequence that the model is representationally ill-defined. The value of system schemas such as circuit diagrams and organization charts is well-known, but system schemas are virtually unknown in such venerable domains as classical mechanics. The heuristic value of system schemas in any domain is immediately obvious to any teacher who instructs students to begin modeling or problem-solving by constructing a system schema. That is a universal solution to the common quandry of how to get started.

Model analysis is concerned with extracting information from a model, such as a physical explanation or an experimental prediction, or merely the answer to a question about the objects that are modeled. For simple linear models this phase can be relatively trivial, but beyond that it may involve solving differential equations or algebraic systems of many variables. In scientific research, model analysis may be a full time job for a theoretical physicist.

Model validation is concerned with assessing the adequacy of the model for characterizing the system/process under investigation. That may involve designing and conducting an experiment to test some prediction from the model. Or it may involve assessing consistency of the model with theoretical results or experimental facts from elsewhere in the scientific community. Students learn that the outcome of this phase must include clear answers to two questions: *What is their model, and how well does it* *work?* They learn gradually what constitutes good scientific answers, including theoretical limitations, sources and estimates of experimental error.

Model deployment consists in adapting a model developed in one context to characterize systems or processes in a totally different context. This serves to sensitize students to the fact that models embody universal structures that can be adapted to modeling in an essentially unlimited number of situations.

The culmination of student modeling activities is reporting and discussing outcomes in a whiteboard session. I believe this is where the deepest student learning takes place, because it stimulates assessing and consolidating the whole experience in recent modeling activities. *Whiteboard sessions* have become a signature feature of the Modeling Method, because they are so flexible and easy to implement, and so effective in supporting rich classroom interactions. Each student team summarizes its model and evidence on a small (2ft × 2.5ft) whiteboard that is easily displayed to the entire class. This serves as a focus for the team's report and ensuing discussion. Comparison of whiteboards from different teams is often productively provocative. The main point is that class discussion is centered on visible symbolic inscriptions that serve as an anchor for shared understanding.

Of course, the pedagogical effectiveness of a whiteboard session depends on the skill and knowledge of the teacher. For implementing the Modeling Method, this is skill in facilitating ***modeling discourse,*** which has two major objectives: The first, as we have already noted, is framing discourse around models and modeling to promote structured understanding of science. The second, more subtle objective is to *engage student physical intuition for tuning to consistency with scientific concepts.* In preparation for that, the modeling workshops sensitize teachers to student intuitions (*aka* misconceptions) about force, as revealed by the FCI. They learn to amplify opportunities for students to articulate their intuitions for public comparison with scientific concepts and evidence. Whiteboard sessions have proved to be an exceptional arena for that. The teachers know that reconciling student intuition with scientific knowledge is a creative act that only students can do for themselves. The best the teacher can do is create the opportunity. From the perspective of Modeling Theory, this is instruction to promote the tuning of p-prims to be consistent with external evidence. This is where the principle of *Form-Content Tension* comes into play. Its implementation is a pedagogical art guided by a little bit of science. As Sherin says [16], "Instruction must nurture and refine intuitive physics, not confront and replace it, or simply build up a new set of frameworks." Physics education researchers David Hammer and Andy Elby [22] emphasize that all students possess powerful *cognitive resources* that can be tapped by a skillful instructor. Their detailed accounts of how to do that have much in common with best practices in Modeling Instruction.

Primacy of modeling over problem solving. According to Modeling Theory, problem solving should be addressed as a special case of modeling and model-based reasoning. The modeling cycle applies equally well to solving artificial textbook problems and significant real world problems of great complexity. Thus, the first step in solving a problem is constructing an explicit model of the situation implicit in conditions of the problem. The next step consists in extracting from the model an answer to the question posed in the problem. This is a special case of model analysis, and an example of model-based reasoning. The final step of "checking the answer" is a special case of model validation.

The modeling method, with its emphasis on coherence and self-consistency of the model, is especially well-suited to detection and correction of ill-posed problems, where the given information is either defective or insufficient. Moreover, students are thrilled when they realize that a single model generates solutions to an unlimited number of problems. Indeed, the Modeling Workshops teach that six basic models suffice to solve almost any mechanics problem in high school physics.

Implications of Modeling Theory for Math Education

The main problem with math education is that the link to physical intuition (the empirical source of mathematical ideas) is seriously degraded, if not broken altogether. *The problem is not with abstraction* in mathematics and mathematical modeling! **Formalization of mathematics** in terms of axioms, rules and algorithms is one of the greatest achievements of mankind, making computer modeling, simulation and data analysis possible, and facilitating construction of objective scientific knowledge

But thinking cannot be reduced to computation, and computers do not understand!! (at least not yet). **Mathematical understanding** requires development of both *physical and mathematical intuition*, which supply the essential *repertoire of mental structures* for constructing robust mathematical meaning. **Physical intuition is cognitively basic**, because it supplies the structural links to bodily experience from which all meaning ultimately derives.

I believe that the best way to address the divorce between math and science at the K-12 level is by integrating math and science instruction, especially in middle school. As pioneered by the Modeling Instruction program, workshops and instructional materials must be developed to enable teachers to enact the necessary reform. School district buy-in will be essential to permit reform. Of course, none of this can happen without substantial commitments and funding.

7. CONCLUSIONS

We have identified and analyzed three fundamental principles of Modeling Theory:

I. Primacy of Models. II. Universal Forms. III. Form-Content Tension.

We have noted their non-trivial implications for the design of curriculum and instruction, with very robust implementation in Modeling Instruction. This is far from exhausting the content and implications of Modeling Theory, so let us dwell briefly on what has been omitted. Many gaps are filled in the literature already cited.

There is much more to be said about *levels of structure* in mental models. At the basic level we have models of *objects*, for cognition is fundamentally *object-oriented*, to use an expression from computer science that probably originates from reflection on intuition. No doubt the central role of objects in cognition derives from *perception*, for perception organizes sensory input into objects situated in an environment. Though objects are cognitively basic, they are not cognitively primitive; they have substructure. The catalog of cognitive primitives evidently includes p-prims and image schemas. These primitives also have structure; they are best described as structured wholes, or *Gestalts*, to use a term that suggests their origin in Gestalt perception. Turning from model substructure to superstructure, we note that mental objects are invariably situated in some mental context or *frame,* sometimes called a *script* or a *scenario* when action or change is modeled. The structure of frames and scripts that is evident in language use provides important clues to the structure of memory. Finally, at the grandest level we note the organization of concepts into conceptual systems.

There is no doubt more to mental models than we have considered so far. We have been concerned mainly with structure that can be represented by rules, as that is the kind of structure in mathematics and objective science. Let's call it *rational structure*, as it may be regarded as the foundation for rational thinking. As mathematician Saunders MacLane [23] asserts, "Mathematics is not concerned with reality but with rule." Mental models also have subjective qualities, such as feeling or emotion, that express significance to the thinker. Emotion is known to play a crucial role in learning and memory. Its relation to rational thinking is yet to be nailed down.

To my mind, the bottom line of Modeling Theory is its implications for teaching and learning. We have seen substantial implications already, and directions for further research are clear. The most important research issues are perhaps in elucidating the mechanisms for creating, changing and maintaining mental structures. Ultimately, this reduces to research in cognitive neuroscience. But to identify brain mechanisms in cognition, it is necessary first to understand at a phenomenological level what cognition consists of.

Finally, before committing to an opinion on Modeling Theory, the reader may wish to ask: Do mental models really exist? Or are they merely figments of a theoretician's imagination? Certainly no one claims that they are directly observable, not even by introspection. The explanation that mental models are not observable because they are located (mostly, at least) in the cognitive unconscious does not answer the question of existence. Cognitive neuroscience even suggests that mental models are epiphenomena at best, for only distributed neural activation patterns occur in the brain.

To forge a scientific answer, comparison of cognitive science with elementary particle theory may be helpful. Physicists are quite confident about the existence of quarks, although quarks are not directly observable even in principle. The reason for the confidence is the explanatory power of quark theory. Likewise, I submit, the existence of mental models hinges on the explanatory power of Modeling Theory. Like quarks, mental models are theoretical constructs, and both exist in the sense that they provide coherence to diverse observations. In other words, both are *invariant objects*, invariant over a range of observations. Is that enough? Could there be more?

8. EPILOGUE: A NEW GENERATION OF MATHEMATICAL TOOLS

The power of mathematics is derived in large part from the design of mathematical tools tools to think with. Like the tools of science and industry, mathematical tools are cultural creations. Many mathematicians would dispute this claim, and they may seem to be supported by standard textbooks, which give a clear impression that mathematics is a complete and permanent edifice that could hardly be improved. As evidence to the contrary, I offer a brief introduction to **geometric algebra**, introducing basic new tools with implications for the whole of mathematics. Though few mathematicians are aware of it, geometric algebra is already a fully developed unified mathematical language for all of physics [24]. Though it has many advanced applications, I concentrate here on implications for mathematics education at the introductory level. First let's consider why that is important.

From the perspective of a practicing scientist, *the mathematics taught in high school and college is fragmented, out of date and inefficient*! The central problem is found in high school geometry. Many schools are dropping the course as irrelevant, and are thus oblivious to the following facts:

- Geometry is the starting place for physical science, the foundation for mathematical modeling in physics and engineering, including the science of measurement in the real world.
- Synthetic methods employed in the standard geometry course are centuries out of date; they are computationally and conceptually inferior to modern methods of analytic geometry, so they are only of marginal interest in real world applications.
- A reformulation of Euclidean geometry with modern vector methods centered on kinematics of particle and rigid body motions will simplify theorems and proofs, and vastly increase applicability to physics and engineering.

We see below how **geometric algebra** can save the day by *unifying high school geometry with algebra and trigonometry* and thereby simplifying and facilitating applications to physics and engineering.

The whole problem boils down to encoding the ***geometric notion of vector as a directed magnitude*** in suitable algebraic form. The standard concepts of *vector addition and scalar multiplication* constitute a partial encoding. What is missing in standard mathematics is a geometrically grounded rule for multiplying vectors. Here is how to fill that gap.

We presume the standard concept of a real vector space and define the geometric product **ab** of vectors by the axioms:

$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc})$	associative,
$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}$	left distributive,
$(\mathbf{b} + \mathbf{c})\mathbf{a} = \mathbf{ba} + \mathbf{ca}$	right distributive,
$\mathbf{a}^2 = a^2$	contraction,

where $a = |\mathbf{a}|$ is a positive scalar called the *magnitude* of **a**, and $|\mathbf{a}| = 0$ implies that $\mathbf{a} = 0$ **.**

These axioms are almost identical to the axioms for ordinary scalar algebra. The main difference is that we need two distributive rules because multiplication is not assumed to be commutative. It is the unassuming contraction rule that sets geometric algebra apart from other associative algebras. Among its many consequences, it implies that the zero scalar and the zero vector are one and the same:

Our main task is to elucidate the geometric meaning of the product **ab**, because that is what gives the algebra its unique power. Historically, the axioms were designed to encode geometric relations [24], so they are by no means arbitrary. We do the reverse here to take advantage of the reader's prior knowledge.

To move quickly to something familiar, we use the geometric product to define the familiar *inner product*:

$$\mathbf{a}\cdot\mathbf{b} \equiv \tfrac{1}{2}(\mathbf{ab}+\mathbf{ba}) = \mathbf{b}\cdot\mathbf{a}\ .$$

To prove that this is identical to the usual Euclidian inner product, we use the axioms to derive the usual *law a of cosines*, thus

$$(\mathbf{a}+\mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 + (\mathbf{ab}+\mathbf{ba}) = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a}\cdot\mathbf{b}\ .$$

It follows that we can write

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta\,,$$

where, of course, θ is the angle between the vectors. This is, in fact, a convenient algebraic definition for the cosine.

Now, to see quickly that we have something genuinely new here, suppose that the vectors are *orthogonal*, which is to say that the inner product vanishes, whence

$$\mathbf{ab} = -\mathbf{ba}\ .$$

Thus, orthogonality is encoded as anticommutivity in geometric algebra. It is easy to prove also that collinearity is encoded as commutativity.

But what is this new entity **ab**? It is neither scalar nor vector. To interpret it, let us first assume that **a** and **b** are orthogonal unit vectors and denote it by a suggestive symbol: $\mathbf{i} = \mathbf{ab}$. Then we can use anticommutivity to prove

$$\mathbf{i}^2 = (\mathbf{ab})^2 = (-\mathbf{ba})(\mathbf{ab}) = -\mathbf{a}^2\mathbf{b}^2 = -1\,.$$

Thus, $\mathbf{i}$ is a truly geometric $\sqrt{-1}$. It is not a scalar, but it can be factored into a product of orthogonal unit vectors, and it can be proved that any such pair of vectors determine the same $\mathbf{i}$. In other words $\mathbf{i}$ is a unique property of a Euclidean plain. To understand this better, we turn to the general case.

It is convenient to define an antisymmetric *outer product* by

$$\mathbf{a}\wedge\mathbf{b} \equiv \tfrac{1}{2}(\mathbf{ab}-\mathbf{ba}) = -\mathbf{b}\wedge\mathbf{a}$$

We can assign a magnitude $|\mathbf{a}\wedge\mathbf{b}|$ to this quantity by

$$|\mathbf{a}\wedge\mathbf{b}|^2 = -(\mathbf{a}\wedge\mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}\cdot\mathbf{b})^2\ .$$

The quantity $\mathbf{a}\wedge\mathbf{b}$ is called a bivector, and it can be interpreted geometrically as an oriented plane segment, as shown in Fig. 7. It differs from the conventional vector cross product $\mathbf{a} \times \mathbf{b}$ in being intrinsic to the plane. Note that the dimension of the vector space has been left unspecified, so all our considerations are quite general.

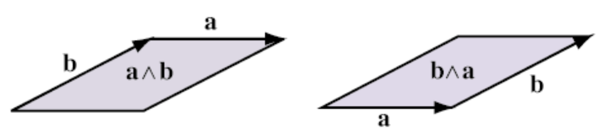

Fig. 7. Bivectors $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ represent plane segments of opposite orientation as specified by a “parallelogram rule” for drawing the segments.

To make connection with trigonometry we can write

$$|\mathbf{a} \wedge \mathbf{b}| = ab \sin\theta \qquad \text{and} \qquad \mathbf{a} \wedge \mathbf{b} = \mathbf{i}|\mathbf{a} \wedge \mathbf{b}| = ab\, \mathbf{i} \sin\theta \,,$$

where $\mathbf{i}$ has been introduced as a *unit oriented area for the plane* containing **a** and **b.** Note that this can be regarded as defining $\sin\theta$ **.**

Now we return to the geometric product note that it has the unique decomposition into symmetric and antisymmetric parts:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

We have noted the geometric meanings of the parts, but what is the meaning of the whole? To relate it to something familiar, we give it a symbol and a trigonometric expression:

$$z \equiv \mathbf{ab} = \lambda U \qquad \text{where} \qquad U = \hat{\mathbf{a}}\hat{\mathbf{b}} = \cos\theta + \mathbf{i}\sin\theta = e^{\mathbf{i}\theta}$$

with $\lambda = ab$. This is the familiar form for a *complex number*, with inner and outer products corresponding to real and imaginary parts. It has all the familiar properties of complex numbers fully integrated with additional properties relating to vectors. In particular, multiplication on any vector **c** in the plane of **i** rotates the vector by angle θ and rescales it by λ, as expressed by

$$z\,\mathbf{c} \equiv \mathbf{abc} = \lambda e^{\mathbf{i}\theta}\mathbf{c} = \mathbf{d} \,.$$

Thus, **the product of two vectors is a complex number, which represents a rotation-dilation in a plane.** As shown in Fig. 8, it can be depicted geometrically as a directed arc (curved arrow), just as a vector is depicted as a straight arrow. See [24] for more details about this interpretation of complex numbers.

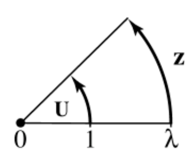

Fig. 8. A complex number depicted as a directed arc.

We have skipped over some mathematical fine points, but the above account suffices to demonstrate that geometric algebra smoothly integrates the algebra of complex numbers with vectors. Thereby, the powerful tool of complex numbers for reasoning about rotations and plane trigonometry becomes available to students from the beginning. Thereby the artificial distinction between real and complex planes is obliterated, and coordinate-free mathematical modeling is enabled.

Geometric algebra extends all this to three dimensions and beyond. For example, it has been applied to reformulate the entire subject of Newtonian mechanics in coordinate-free form [25]. This includes computation of rotations and rotational dynamics without matrices. Moreover, all this has been generalized to computations in linear algebra without matrices and applications to many other domains of mathematics.

I will not speculate here on the prospects for incorporating geometric algebra into the mathematics curriculum.

REFERENCES

[1] Arnold, V.I. (1997), *On Teaching Mathematics.* The long text of his hard-hitting address and traces of its aftershocks are easily found by googling his name.

[2] Kline, M. (1980), *Mathematics, the loss of certainty.* Oxford U. Press: New, York.

[3] Hestenes, D. (2007). Notes for a Modeling Theory of Science. Cognition and Physics Education. In A.L. Ellermeijer (ed.) *Modelling in Physics and Physics Education.* Available at the Modeling Instruction website*:* http://modeling.asu.edu.

[4] Cassirer, E. *(*1953), *The Philosophy of Symbolic Forms.* Yale U. Press: New Haven. Four volumes.

[5] Evans, V. & Green, M. *(*2006), *Cognitive Linguistics, an Introduction.* Lawrence Erlbaum: London.

[6] Croft, W. & Cruse, D.A. *(*2004), *Cognitive Linguistics.* Cambridge U Press: Cambridge.

[7] Kant, I. (1781, 1787). *Critique of Pure Reason* (Palgrave Macmillan, 2003). German editions translated by Norman Kemp Smith (1929).

[8] Lakoff, G. & Johnson, M. (1999), *Philosophy in the Flesh.* Basic Books: New York.

[9] Goldman, S., Graesser, A. & van den Broek, P, *(*2001), *Narrative Comprehension, Causality and Coherence.* Lawrence Earlbaum: London.

[10] Shapiro, S. (1997), *Philosophy of Mathematics, structure and ontology.* Oxford U Press: New York.

[11] Lakoff, G. & Núñez, R. (2000), *Where mathematics comes from.* (Basic Books, New York).

[12] Hadamard, J. (1945). *An Essay on the Psychology of Invention in the Mathematical Field.* Dover: New York.

[13] Cassam, Q. (2007), *the possibility of knowledge.* Oxford U Press: Oxford.

[14] Hertz, H, (1956), *The Principles of Mechanics.* Dover: New York.

[15] diSessa, A. (1993), Toward an Epistemology of Physics, *Cognition & Instruction* **10:** 105-225.

[16] Sherin, B. (2006). Common sense clarified: Intuitive knowledge and its role in physics expertise. *Journal of Research in Science Teaching* **43**: 535–555.

[17] Sherin, B. (2001). How students understand physics equations. *Cognitive Science* **19**: 479541.

[18] Schwartz, D.L., Martin, T., & Pfaffman, J. (2005). How Mathematics Propels the Development of Physical Knowledge. *Journal of Cognition and Development* **6:** 65-88.

[19] Schwartz, D.L. & Moore, J.L (1998). The role of mathematics in explaining the material world: Mental models for proportional reasoning. *Cognitive Science* **22** 471-516.

[20] Romer, R. (1993). Reading equations and confronting the phenomena; the delights and dilemmas of physics teaching. *American Journal of Physics*. **61**: 128-124.

[21] Hestenes, D. (1992). Modeling Games in the Newtonian World, *American Journal of Physics* **60**: 732-748.

[22] Hammer, D. & Elby, A. (3003), Tapping Epistemological Resources for Learning Physics. *Journal of the Learning Sciences.* **12**: 53-90.

[23] MacLane, S. (1986), *Mathematics, Form and Function.* Springer-Verlag: New York.

[24] D. Hestenes (2003), Oersted Medal Lecture 2002: Reforming the Mathematical Language of Physics, *American Journal of Physics* **71**:104-121. Available at the Geometric Calculus website: <http://modelingnts.la.asu.edu>

[25] Hestenes, D. (1999), *New Foundations for Classical Mechanics*, (Reidel: Dordrecht/Boston, 1986), Second Edition, Kluwer: Dordrecht/Boston.