In: Hermann Gunther Grasmann (1809-1877): Visionary Mathematician, Scientist and Neohumanist Scholar, 1996 (Gert Schubring, Ed.), Kluwer Academic Publishers, Dordrecht/Boston 191–201.

GRASSMANN'S VISION

David Hestenes

Abstract. Hermann Grassmann is cast as a pivotal figure in the historical development of a *universal geometric calculus* for mathematics and physics which continues to this day. He formulated most of the basic ideas and, to a remarkable extent, anticipates later developments. His influence is far more potent and pervasive than generally recognized.

After nearly a century on the brink of obscurity, Hermann Grassmann is widely recognized as the originator of *Grassmann algebra*, an indispensable tool in modern mathematics. Still, in conception and applications, conventional renditions of his *exterior algebra* fall far short of Grassmann's original vision. A fuller realization of his vision is found in other mathematical developments to which his name is not ordinarily attached. This Sesquicentennial Celebration of Grassmann's great book, *Die Lineale Ausdehnungslehre* [1], provides the opportunity for a renewed articulation and assessment of Grassmann's vision.

Grassmann is a pivotal figure in the historical evolution of a mathematical language to characterize human understanding of the physical world. Since quantitative concepts of space and time are fundamental to that understanding, the language is fundamentally geometrical and can best be described as a *geometric calculus.* For the most part the evolution of geometric calculus has been tacit and piecemeal, with many individuals achieving isolated results in response to isolated problems. Grassmann is pivotal in this historical process because he made it explicit and programmatic. In no uncertain terms, he declared the goal of his research on *extension theory* [1] as no less than the creation of a *universal instrument for geometric research.* Leibniz had clearly articulated that goal long before and blessed it with the name "geometric calculus." However, Grassmann was the first to see clearly how the goal can be reached, and he devoted the whole of his mathematical life to the journey. When Möbius created the opportunity for Grassmann to align his own vision with Leibniz's, Grassmann responded eagerly with an elaboration of his extension theory to show how their common vision can be realized [2]. Unfortunately, this seminal work failed to attract the recognition and attention he deserved, so Grassmann had to continue his mathematical journey alone, with only an occasional mathematician — Möbius, Hankel, Clebsch — joining him for a few steps along the way.

Most of Grassmann's mathematical discoveries were duplicated independently by others as isolated results. It is fair to say, for example, that Grassmann laid the foundation for linear algebra by himself [3]. Yet the standard linear algebra of today grew up without his contribution. The frequent duplication of Grassmann's discoveries is not a mark of limited originality but rather a sign that Grassmann was keenly attuned to a powerful thematic force driving mathematical development, namely, the subtle interplay between geometry and algebra. What sets Grassmann ahead of other creative mathematicians is his systemic vision of a universal geometric calculus. This vision marks him as one of the great conceptual synthesizers of all time. Accordingly, the chief presentation of that vision, *Die Lineale Ausdehnungslehre*, deserves a place alongside Euclid's *Elements* and Descartes' *Geometrié* in the library of immortal mathematical masterworks.

Considering the immense power and fertility of Grassmann's vision, we are bound to ask why due recognition has been so long in coming. Several answers have been suggested by commentators and historians. First, it is averred that the original *Ausdehnungslehre* was written in an impenetrable philosophical style. But Grassmann confined his philosophical musings to the first chapter so they could be skipped by those who want to get directly to business. Second, it is suggested that Grassmann was ahead of his time. He was certainly the first person to deal with abstract, multidimensional algebra. But if that was a barrier to understanding during his lifetime, it should have been lesser in subsequent generations. Third, it is noted that Grassmann's unsystematic presentation mixes basic ideas with applications and so places a heavy burden on the reader to disentangle them. But Grassmann's synopsis of the *Ausdehnungslehre* [4], written at the behest of a befuddled admirer, did not help much to ease the burden. Moreover, more systematic presentations of Grassmann's calculus by distinguished mathematicians, most notably Peano in 1888 and Whitehead in 1898, also failed to inspire the mathematical community at large.

Grassmann himself identified a more critical condition for comprehending his vision, namely *immersion* in his conceptual system. That system is, after all, a rich mathematical language, so it takes years to develop the proficiency that opens up mathematical insights. Having acquired such proficiency, the superiority of his system in applications throughout geometry and physics was obvious to Grassmann, so it was a great frustration to him that others, even distinguished mathematicians, were unable to recognize this. It was easy for experts to dismiss Grassmann's work as "old wine in new bottles" when he was unable to show them applications that could not be handled with techniques they already knew. Grassmann thus suffered from a credibility gap that prevented experts from investing the time necessary to acquire his perspective.

There is another, more profound reason for the muted impact of Grassmann's extension theory. Something critical was missing. As Grassmann admitted in the preface to his *Ausdehnungslehre* of 1862 [6]. "I am aware that the form which I have given the science is imperfect." And he went on to say "there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into a living communication with contemporary developments."

The purpose of this paper is to point out that the missing ingredient in Grassmann's theory has been found and put in place to create a geometric calculus that fully realizes his vision. Indeed, the seeds for this advance were already present in Grassmann's work. Moreover, Grassmann's bold prophecy is in the process of fulfillment even today, as his insights are revived to enrich modern mathematics and physics.

I. Completing the Vision

In the Foreword to his *Ausdehnungslehre* of 1844, Grassmann outlined some ideas to be incorporated in a second volume to "complete the work." This included the invention of a new kind of product, the essentials of which can be described as follows. He defined the *quotient* $\mathbf{a}/\mathbf{b} = \mathbf{a}\mathbf{b}^{-1}$ of vectors $\mathbf{a}$ and $\mathbf{b}$ by writing

$$(\mathbf{a}/\mathbf{b})\mathbf{b} = \mathbf{a}. \tag{1}$$

He noted that, if $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude, then (1) implies that $\mathbf{a}/\mathbf{b}$ is an operator which rotates $\mathbf{b}$ into $\mathbf{a}$. He expressed this with the exponential form

$$e^{\angle \mathbf{ab}} = \mathbf{a}/\mathbf{b}, \tag{2}$$

where $\angle \mathbf{ab}$ denotes the angle between $\mathbf{a}$ and $\mathbf{b}$. It follows then that $(\mathbf{a}/\mathbf{b})^2$ is a rotation through twice the angle. In particular, if $\mathbf{a}/\mathbf{b}$ is a right angle rotation, then $(\mathbf{a}/\mathbf{b})^2\mathbf{b} = -\mathbf{b}$, so $(\mathbf{a}/\mathbf{b})^2 = -1$ and

$$\mathbf{a}/\mathbf{b} = \sqrt{-1}\,. \tag{3}$$

If we write $\angle \mathbf{ab} = \mathbf{i}x$, where $\mathbf{i} = \sqrt{-1}$ and x is the radian measure of the angle, then

$$e^{\mathbf{i}x} = \cos x + \mathbf{i} \sin x \tag{4}$$

has a purely geometrical significance denoting the rotation through an arbitrary angle.

Grassmann concludes: "From this all imaginary expressions now acquire a purely geometric meaning, and can be described by geometric constructions.... it is likewise now evident how, according to the meaning of the imaginaries thus discovered, one can derive the laws of analysis in the plane; however it is not possible to derive the laws for space as well by means of imaginaries. In addition there are general difficulties in considering the angle in space, for the solution of which I have not yet had sufficient leisure."

That is the last we hear about this interpretation of the imaginary unit from Grassmann. He never revived and completed the argument to handle general rotations in space and develop his new product into a complete algebraic system. However, a brilliant young English mathematician, William Kingdom Clifford, did just that. The resulting new algebraic system is known as *Clifford algebra* in mathematical circles today.

Ironically, Clifford is seldom mentioned in accounts of Grassmann's influence on other mathematicians, though it may be through Clifford algebra that Grassmann's ideas exert their most profound influence today. We have no direct knowledge that Clifford was influenced by Grassmann's little argument above, but it could hardly be otherwise, because the argument appeared right at the beginning and nothing else in Grassmann's corpus is so obviously pertinent. Indeed, Clifford made no great claim to originality, referring to his algebra as a mere application of Grassmann's extensive algebra [7]. Unfortunately, his life was too brief for him to communicate the dimensions of his debt to Grassmann.

Clifford was surely assisted in his adaptation of Grassmann's ideas by his mastery of Hamilton's quaternions. Ironically again, Grassmann was led along the same path in the last years of his life by his endeavor to fit quaternions into extension theory [8]. He was induced to define the *central product* $\mathbf{ab}$ of vectors $\mathbf{a}$ and $\mathbf{b}$ by writing

$$\mathbf{ab} = \lambda[\mathbf{a}|\mathbf{b}] + \mu[\mathbf{ab}], \tag{5}$$

where $[\mathbf{a}|\mathbf{b}]$ is the *inner product*, $[\mathbf{ab}]$ is the *outer product*, and λ, μ are arbitrary nonzero constants. With $\lambda = \mu = 1$, we have the alternative notation

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}. \tag{6}$$

This is essentially the basic product of Clifford algebra, and it was published before Clifford's paper [7]. Thus, Grassmann could be credited with the invention of Clifford algebra. However, he failed to investigate the central product with the verve that he bestowed on the inner and outer products in bygone years. Had he done so, he would have been pleasantly surprised to find that quaternions drop out without resorting to the deformation of the central product (5) that he employed in his paper [8].

Grassmann also failed to notice that the product in (6) can be identified with the quotient in (2). Then, for $\mathbf{a}^2 = \mathbf{b}^2 = 1$, we can write (2) in the form

$$\mathbf{ab} = e^{\mathbf{i}x}. \tag{7}$$

Using (2) and (6) to expand the right and left sides of (7), we infer that $\mathbf{a} \cdot \mathbf{b} = \cos x$ and, more important,

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{i} \sin x. \tag{8}$$

This tells us immediately that the *unit imaginary* $\mathbf{i}$ must be interpreted as the unit bivector for the plane containing $\mathbf{a}$ and $\mathbf{b}$, something that Grassmann never realized.

There is another easy inference that would have pleased Grassmann mightily, because it reveals a hidden unity and simplicity in his system which he had not anticipated. Using the symmetry of the inner product and the skew symmetry of the outer product, we infer from (6) that

$$\mathbf{ba} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b}. \tag{9}$$

Then, from (6) and (9) we can deduce

$$\mathbf{a} \cdot \mathbf{b} = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) \tag{10}$$

and

$$\mathbf{a} \wedge \mathbf{b} = \tfrac{1}{2}(\mathbf{ab} - \mathbf{ba}). \tag{11}$$

This shows that the inner and outer products can be defined in terms of the central product, thus reducing three products to one. Full details of this reduction and simplification of extension theory are given in [9].

The insight that all multiplicative aspects of extension theory can be reduced to the single central product leads to the conclusion that Clifford algebra is the proper completion of Grassmann's vision for a universal geometric calculus. Ample justification for this conclusion comes from the manifold applications of Clifford algebra mentioned below. Recognizing the geometric origin of his algebra, Clifford called it *geometric algebra*. Considering the universality of this algebra and the fact that so many others besides Grassmann and Clifford have contributed to its development as a mathematical language, it should be regarded as common intellectual property of the mathematical community, so it should not be tagged with the name of a single individual. As no other mathematical system deserves the title more, Clifford's choice should be adopted and, on occasion, expanded to *universal geometric algebra* for emphasis. For the same reasons, the name *geometric product* is to be preferred in place of Grassmann's "central product."

As counterpoint to these unifying developments, we note a final irony in Grassmann's treatment of quaternions in his dismissal of Hamilton's ideas without due credit, just as his own ideas had been summarily dismissed by other mathematicians before. It is likely that Grassmann did not have the opportunity to study Hamilton's work first hand, so he learned about it from secondary sources. That only deepens the irony in his refusal to admit that Hamilton has something to teach him. He could, at least, have graciously acknowledged that the crucial idea of adding a scalar to a bivector in (5) came from Hamilton, for he himself had never taken this step before. Indeed, he had explicitly warned against such mixing of quantities of different kind.

II. Source of the Vision

The incredible power and versatility of Grassmann's approach derives from the fact that he systematically incorporated geometric interpretations into the design of his algebraic syntax; then he abstracted the algebraic structure from the geometric interpretation for analysis as a formal algebraic system.

Grassmann introduced two complementary geometric interpretations for vectors and their outer products. Let us refer to them as the *direct* and the *projective* interpretations to keep them distinct. One difficulty in understanding Grassmann's work is the informal fluency with which he moves from one interpretation to the other. His synopsis [4] is helpful by bringing the two interpretations together and comparing them. Recently, it has been realized that *geometric algebra* provides a surprising formal connection between the interpretations [10, 11, 12] and thus clarifies their complementary roles. To see that we need to review Grassmann's perspective.

We begin with the modern concept of a *vector* as an abstract element of a vector space. The concept of a vector space had not been formalized in Grassmann's time, though it was certainly implicit in his work. To coordinate the two geometric interpretations, we assign them to vectors in different spaces which are distinguished by using *boldface* to denote vectors with the direct interpretation and *italics* for vectors with a projective interpretation.

Under the *direct interpretation* each vector $\mathbf{a}$ represents a directed line segment, and the outer product $\mathbf{a} \wedge \mathbf{b}$ represents a directed plane segment. This interpretation is thoroughly discussed and applied to classical mechanics in ref. [9] using the full geometric algebra.

Under the *projective interpretation* each vector a represents a geometric point. The outer product $a \wedge b$ represents a directed line segment determined by points a and b, while $a \wedge b \wedge c$ represents a directed plane segment determined by three points. This interpretation is thoroughly discussed and applied to projective geometry in ref. [10]. which also explains the advantages of the full geometric algebra in this domain.

The two different interpretations correspond to two different representations of geometric concepts. The relation between the two is elegantly characterized by a *projective split* in geometric algebra, as explained in ref. [11] and also in [12], which is a synopsis of [10] and

[11]. The projective split relates each vector $\mathbf{a}$ in an n-dimensional vector space $\mathcal{V}_n$ to a vector a in an $(n+1)$-dimensional vector space $\mathcal{V}_{n+1}$ by the equation

$$a\mathbf{e}_0 = a \cdot \mathbf{e}_0 + a \wedge \mathbf{e}_0 = a_0(1 + \mathbf{a}) \,, \tag{12a}$$

where $\mathbf{e}_0$ is a distinguished vector in $\mathcal{V}_{n+1}$, and

$$a_0 = a \cdot \mathbf{e}_0 \,, \tag{12b}$$

$$\mathbf{a} = \frac{a \wedge \mathbf{e}_0}{a \cdot \mathbf{e}_0} \,. \tag{12c}$$

According to (12c), vectors in $\mathcal{V}_n$ correspond to bivectors of $\mathcal{V}_{n+1}$ with a common factor $\mathbf{e}_0$. This is a projective relation, because (12c) shows that $\mathbf{a}$ is unaffected if a is replaced by a nonzero scalar multiple of a. The projective split amounts to a mechanism for introducing *homogeneous coordinates*. It is more powerful than the conventional approach to homogeneous coordinates, however, because it prepares the way for drawing on the computational and conceptual advantages of geometric algebra. Since there is insufficient space here to demonstrate those advantages, the reader is left to consult the references.

Grassmann made much ado about separating *extension theory* from *space theory*, in other words, separating formal algebraic structures from geometric interpretation. He was ahead of his time in regarding geometry (space theory) as an empirical discipline derived from human perception and therefore limited to spaces of no more than three dimensions. Extension theory formalizes space theory, and Grassmann emphasized that geometric interpretation is essential to its applications in physics and geometry. On the other hand, he argues that separation from space theory enables the cultivation of extension theory as an abstract science and admits generalization to higher dimensions. He could as well have added that geometric interpretations can also be generalized to give insight to algebraic structures in higher dimension.

One clear benefit of separating geometric algebra from geometric interpretation is that a single algebraic entity can be assigned many different geometric interpretations. We have already seen, for example, that vectors can be assigned either direct or projective interpretations, and this does not exhaust the possibilities. However, abstraction has its dangers, as the following important historical example serves to show.

After Clifford algebra was invented by Clifford, it was cultivated as an abstract discipline for the better part of a century without reference to its geometrical roots. Clifford algebra thus became a minor mathematical subspecialty — just one more example of an algebra among many others. In regard to applications, it was almost sterile, with representations of the orthogonal group as the only example of much interest.

The situation has changed drastically since 1966 when, for the purpose of application to physics, Grassmann's direct interpretation was incorporated into Clifford algebra and a variant of the projective split (now called the *spacetime split*) was found to simplify and clarify relativistic physics [13]. This has reinvigorated Clifford algebra and fueled a growing realization that it provides the structural basis for a truly universal geometric algebra.

III. Realizing the Vision

Realizing Grassmann's vision of a universal geometric calculus to serve as a unified language for mathematics and physics is necessarily a community enterprise. Though Grassmann and Clifford have supplied the language with a suitable syntax, enrichment of the language with a broad spectrum of applications is a vast undertaking drawing on the intellectual resources of the entire scientific community. Grassmann understood this perfectly, so he devoted considerable effort to developing applications in geometry and physics, primarily mechanics and electrodynamics.

The main concern of Grassmann's scientific papers is the mathematical form of physical equations. Much of his work on mechanics is standard fare in textbooks today, with only minor differences in notation, though direct influence of Grassmann on textbooks is unlikely. The only mechanics problem of any complexity that Grassmann addressed is the "theory of the tides." While that work has no special scientific importance, a reformulation of its vectorial perturbation approach to tidal theory in modern terms may have pedagogical value. The main deficiency in Grassmann's approach to tidal theory and the rest of mechanics was in his mathematical treatment of rotations and rotational dynamics. Though Grassmann attacked this problem with great ingenuity, the ideal solution escaped him, because it required the geometric product. Grassmann's grand vision for mechanics has recently been utterly fulfilled with a complete reformulation of the subject in terms of geometric algebra [9]. As Grassmann desired, the treatment is completely coordinate-free, it employs his direct geometric interpretation, and it incorporates all of his keen algebraic devices.

At this point I must confess my personal interest and apologize for the extensive self-citations in a tribute to someone else. The fact is that most of my professional life has been devoted to the same grand theme as Grassmann's, though I came to realize this only recently. My work could fairly be described as restoring Grassmann's original insights to Clifford algebra and systematically developing it into a universal language through extensive applications in mathematics and physics. I was guided by Grassmann's ideas from the beginning and referred to them frequently in my published work. However, everything I knew about Grassmann came from secondary sources, so it suffered from the limitations and distortions of such filtering. It was only in the last few years that I learned about the incredible confluence of my own vision with Grassmann's from the translation of Grassmann's work by Lloyd Kannenberg. I was amused to note that I had independently rederived practically every algebraic identity in Grassmann's work (especially in [14]). This illustrates how a shared vision can stimulate parallel discoveries. That occurs frequently in mathematics, and, as I have emphasized before [15], it is especially significant in the history of geometric algebra because of its universal importance.

In my own work on geometric calculus, I have enjoyed the huge advantage over Grassmann of more than a century of developments in mathematics and physics. Even so, I find little in Grassmann's mathematical work that is outdated, and I am confident in asserting that Grassmann's optimistic vision of the future for geometric calculus has been fully vindicated. The references cited herein demonstrate in detail that geometric calculus embraces a greater range of mathematics than any other mathematical system, including linear and multilinear algebra, projective geometry, distance geometry [16], calculus on manifolds [17], hypercomplex function theory, differential geometry, Lie groups and Lie algebras [18].

Geometric calculus has also fulfilled Grassmann's vision of a universal language for physics. Besides integrating the mathematical formulations of mechanics [9, 19], relativity and electrodynamics [13], it has revealed a geometric basis for complex numbers in quantum mechanics [20, 21, 22].

Leading into the twentieth century, Grassmann's algebra was often cast as a competitor with quaternions and vector calculus for the role of a language for physics. Vector calculus was victorious, owing principally to Oliver Heaviside's slick formulation of Maxwell's equations with applications to electromagnetic wave propagation. The victory was only temporary, however. With the completion of Grassmann's vision by geometric calculus, we can now see Heaviside's vectorial equations and Hamilton's quaternions as components of a single, more powerful system. Indeed, the geometric calculus reduces the four Maxwell equations of Heaviside to the single equation [13, 15]

$$\partial F = J, \tag{13}$$

where J is the spacetime charge current density, F is the complete electromagnetic field (without separation into electric and magnetic parts), and ∂ is the vector derivative with respect to a spacetime point. With the choice of an inertial reference frame, (13) can be split into the four equations of Heaviside [13]. However, in many applications it is simpler and more informative to solve (13) directly. Equation (13) is typical of the simplifications that geometric calculus brings to every branch of physics. This justifies its claim to be a universal language for physics.

The fusion of Grassmann algebra with Clifford algebra has been the subject of some debate [23]. The chief point of contention has been the concept of *duality*. Dieudonné summarizes the dominant "modern view" in [5] and declares it superior to Grassmann's. In fact, they are just different, and Grassmann's has distinct advantages which are made more evident by using the full geometric algebra [10]. On the other hand, the modern view has unnecessary limitations which can be eliminated with geometric algebra. It works with a "dual pair" of vector spaces, each with its own exterior algebra. As shown in [18], a far more powerful algebraic structure is created at no cost by joining the dual pair into a single vector space with twice the dimension of each and then incorporating the duality relations into the geometric algebra of that space. The remarkable result is that *every* linear transformation on the base space can be represented in this algebra as a monomial product of vectors. This promises to open a new chapter in linear algebra.

IV. Recognizing the Vision

An adequate history of geometric calculus remains to be written. It calls for a historian steeped in Grassmann's vision who can recognize how the vision is played out in the various branches of mathematics and physics. We have seen how easy it is to misconstrue or overlook important historical threads. Even the central theme — one of the great themes of cultural history — has been consistently overlooked by mathematicians and historians. The time is ripe for a historian with the insight and dedication to flesh out the whole story.

REFERENCES

[1] H. Grassmann, Linear Extension Theory (*Die Lineale Ausdehnungslehre*), translated by L. C. Kannenberg in **The "Ausdehnungslehre" of 1844 and Other Works**, Open Court Publ. (in press, 1994).

[2] H. Grassmann, Geometric Analysis, translated by L. C. Kannenberg in **The "Ausdehnungslehre" of 1844 and Other Works,** Open Court Publ. (in press, 1994).

[3] D. Fearnley-Sander, Hermann Grassmann and the Creation of Linear Algebra, *American Mathematical Monthly*, 809–817 (Dec. 1979).

[4] H. Grassmann, Survey of the Essentials of Extension Theory, translated by L. C. Kannenberg in **The "Ausdehnungslehre" of 1844 and Other Works**, Open Court Publ. (in press, 1994).

[5] J. Dieudonné, The Tragedy of Grassmann, *Linear and Multilinear Algebra* **8**, 1–14 (1979).

[6] Translation by M. Crowe in **A History of Vector Analysis**, Notre Dame (1967).

[7] W. K. Clifford, Applications of Grassmann's Extensive Algebra, *Am. J. Math.* **1**, 350–358 (1878).

[8] H. Grassmann, The Position of Hamiltonian Quaternions in Extension Theory, H. Grassmann, Survey of the Essentials of Extension Theory, translated by L. C. Kannenberg in **The "Ausdehnungslehre" of 1844 and Other Works**, Open Court Publ. (in press, 1994).

[9] D. Hestenes, **New Foundations for Classical Mechanics**, G. Reidel Publ. Co., Dordrecht/Boston (1985), paperback (1987). Third printing 1993.

[10] D. Hestenes and R. Ziegler, Projective Geometry with Clifford Algebra, *Acta Applicandae Mathematicae* **23**, 25–63 (1991).

[11] D. Hestenes, The Design of Linear Algebra and Geometry, *Acta Applicandae Mathematicae* **23**, 65–93 (1991).

[12] D. Hestenes, Universal Geometric Algebra, *Simon Stevin* **62**, 253–274 (1988).

[13] D. Hestenes, **Space-Time Algebra**, Gordon & Breach, New York, (1966).

[14] D. Hestenes and G. Sobczyk, **CLIFFORD ALGEBRA to GEOMETRIC CALCULUS, A Unified Language for Mathematics and Physics**, G. Reidel Publ. Co., Dordrecht/Boston (1984), paperback (1985). Third printing 1992.

[15] D. Hestenes, A Unified Language for Mathematics and Physics. In J.S.R. Chisholm/A.K. Common (eds.), **Clifford Algebras and their Applications in Mathematical Physics.** Reidel, Dordrecht/Boston (1986), p. 1–23.

[16] A. Dress & T. Havel, Distance Geometry and Geometric Algebra, *Foundations of Physics* **23**, 1357–1374.

[17] D. Hestenes, Differential Forms in Geometric Calculus. In F. Brackx *et al* (eds.), **Clifford Algebras and their Applications in Mathematical Physics**, Kluwer: Dordrecht/Boston (1993), p. 269–285.

[18] C. Doran, D. Hestenes, F. Sommen & N. Van Acker, Lie Groups as Spin Groups, *Journal of Mathematical Physics* **34**, 3642–3669 (1993).

[19] D. Hestenes, Hamiltonian Mechanics with Geometric Calculus. In Z. Oziewicz, A. Borowiec & B. Jancewicz (eds.), **Spinors, Twistors and Clifford Algebras**. Kluwer: Dordrecht/Boston (1993), p. 203–214.

[20] D. Hestenes, Clifford Algebra and the Interpretation of Quantum Mechanics. In J.S.R. Chisholm/A.K. Common (eds.), **Clifford Algebras and their Applications in Mathematical Physics**. Reidel, Dordrecht/Boston (1986), p. 321–346.

[21] D. Hestenes, The Kinematic Origin of Complex Wave Functions. In W. T. Grandy & P. W. Milloni (eds.), **Physics and Probability: Essays in Honor of Edwin T. Jaynes**, Cambridge U. Press, Cambridge (1993), p. 153–160.

[22] S. Gull, A. Lasenby & C. Doran, Imaginary Numbers are not Real – The Geometric Algebra of Spacetime, *Foundations of Physics* **23**, 1175–1201.

[23] D. Hestenes, Mathematical Viruses. In A. Micali, R. Boudet, J. Helmstetter (eds.), **Clifford Algebras and their Applications in Mathematical Physics**. Kluwer, Dordrecht/Boston (1991), p. 3–16.