Grassmann's Legacy

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In a previous conference honouring Hermann Grassmann's profound intellectual contributions (Schubring 1996), I cast him as a central figure in the historical development of a *universal geometric calculus* for mathematics and physics (Hestenes 1996). Sixteen years later I am here to report that impressive new applications in this tradition are rapidly developing in computer science and robotics as well as physics and mathematics. Especially noteworthy is the emergence of *Conformal Geometric Algebra* as an ideal tool for computational geometry, as it fulfils at last one of Grassmann's grandest goals and confirms the prescience of his mathematical insight. Geometric Calculus has finally reached sufficient maturity to serve as a comprehensive geometric language for the whole community of scientists, mathematicians and engineers. Moreover, its simplicity recommends it as a tool for reforming high school mathematics and physics, as Grassmann had envisioned.

My purpose here is threefold: to extend my previous account of Grassmann's pivotal role in the evolution of Geometric Algebra to place him in a broad historical context; to survey landmarks in the recent development of Geometric Calculus that demonstrate its current vigour and broad applicability; to explain precisely what extensions of Grassmann's system were needed to meet his ambitious goals.

Evolution of Geometric Algebra and Calculus

For present purposes, 'Geometric Algebra' and 'Geometric Calculus' can be regarded as synonymous, with algebra regarded as a tool for calculation. (I use capitals to emphasize specific meanings for the terms 'geometric algebra and calculus' employed here.) Technically speaking, 'Geometric Calculus' is the broader term, referring to an extension of 'Geometric Algebra' to differentiation and integration, including differential geometry and differential forms (Hestenes and Sobczyk 1984). However, our main interest here is the underlying algebraic structure.

A family tree of major steps in the evolution of Geometric Algebra is laid out in Fig. 1. We have space here for only a few comments about it. The main line proceeds directly from Euclid through Grassmann and Clifford to the beginning of Geometric Calculus (Hestenes 1966). Perhaps Descartes should be included in this sequence, because analytic geometry was surely a crucial input to Grassmann's thinking, though Grassmann drew directly on Euclid to create a coordinate-free algebra of geometric concepts.

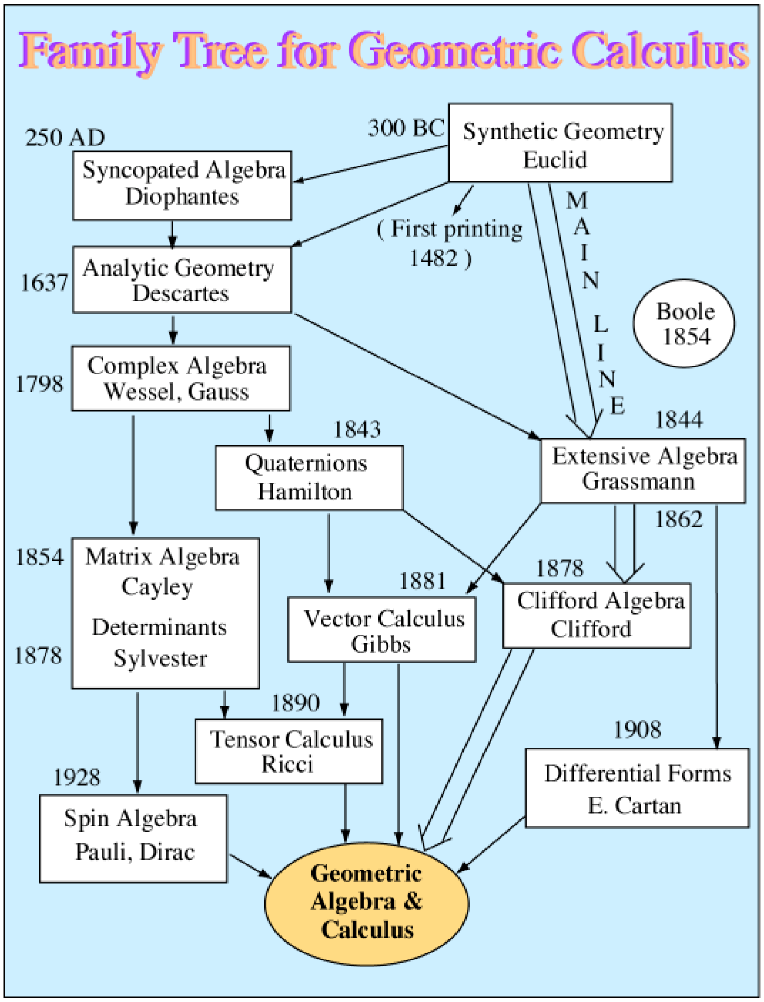

Figure 1

I have included Boole in an unconnected bubble in Fig. 1 as a reminder that extracting logical structure from natural language has much in common with Grassmann's program to put synthetic geometry in algebraic form.

A branch of Grassmann followers including Peano, Whitehead and Forder has been pruned from the Family Tree, because they did not significantly advance Grassmann's agenda or contribute to the emergence of Geometric Algebra.

Ironically, W. K. Clifford (1878), the mathematician exhibiting the deepest understanding of Grassmann's system and advancing it in a major way, is seldom mentioned as a follower of Grassmann in historical accounts, though Clifford himself could not have been more explicit or emphatic in his claim to be following Grassmann in developing what he called "*geometric algebra!*" Mathematicians too have overlooked Clifford's link to Grassmann, and, to this day, treat Clifford algebra as a completely separate algebraic system.

Elie Cartan (1922) incorporated Grassmann's outer product into his calculus of *differential forms.* Though it put Grassmann's name into the mathematics mainstream, it so diluted his ideas that Engel called it "Cartanized Grassmann." In another irony, Cartan (1968) also employed a matrix form of Clifford algebra in his "theory of spinors," but he failed to recognize its relation to Grassmann algebra and differential forms.

In physics, Clifford algebra was rediscovered in the matrix algebras of Pauli and Dirac where it plays an essential role in quantum mechanics. Finally, in 1959 the many threads in Fig. 1 converged to a rebirth of Grassmann's vision of a universal geometric algebra with powerful applications to physics. The midwife of the rebirth was a set of lecture notes by Marcel Riesz (1958). Let me describe briefly what happened.

I was a graduate student in 1959 intensively studying the alternative mathematical systems used in physics, including the Feynmann trace calculus in quantum electrodynamics and tensor calculus in general relativity. I was lucky to get a thorough introduction to differential forms, including its intuitive origins, in a course on differential geometry by Barret O'Neil, as there was no good book on the subject in English at the time. I was consciously concerned with questions relating the structure of these mathematical systems to the structure of the physical world. One day Riesz's notes appeared on the new-book shelf of the UCLA library. The impact on me was immediate and striking! By the time I was half way through the first chapter I was convinced that Clifford algebra was the key to unifying mathematical physics. During the next few years I worked out the framework for a fully geometric unification. The result was published in my book *Space Time Algebra* (1966).

Fortunately, my book was widely distributed, and it helped me establish many fruitful contacts throughout the world in subsequent years. However, I believe its impact would have dissipated had I not followed it up with years of further research, lectures and publications. Also, I believe that the significance of Riesz's notes would have remained unrecognized without the citation in my book, which eventually led to publication (Riesz 1993).

Though the book launched me on a program to unify mathematical physics, I refrained from proclaiming the product as a *universal Geometric Algebra and Calculus* until subsequent research convinced me that was fully justified. I was well aware that its roots were in the work of Grassmann, but it was not until the English translations of the *Ausdehnungslehre* (Grassmann 1995, 2000) by Lloyd Kannenberg that I realized how deep those roots were.

Independently, I had rederived most of Grassmann's algebraic identities (as have others), for they are universal algebraic truths. That was better than getting results directly from Grassmann, for it enhanced my appreciation of his groundbreaking work and helped me see it from a different perspective. All the same, Grassmann still has much to teach us.

Recent Developments in Geometric Algebra

The Geometric Algebra/Calculus bubble in Fig. 1 is unpacked in Fig. 2 to outline major developments including a recent surge in applications. Let me explain the significance of each box with reference to a key publication from which the literature can be traced. The reader is invited to correlate my explanations with the figure as I proceed.

To fulfil the promise of Geometric Calculus as a comprehensive mathematical language for all of physics, my book *Space Time Algebra* launched me along *three main lines of research and development* that were clearly demarcated and consolidated within the next two decades.

The first line was a straightforward reformulation of classical physics in terms of geometric algebra. It produced the first comprehensive coordinate-free treatment of Newtonian mechanics, including rotational dynamics (Hestenes 1985). Both Grassmann and Clifford had a similar goal, but comparison with their work shows what a difference a century of science can make. A similar reformulation of classical electrodynamics was equally straightforward and enlightening (Baylis 1999).

The second line of research emerged from reformulating the Dirac equation in terms of Geometric Algebra. This revealed a hidden geometric structure in quantum mechanics, including a hitherto unrecognized geometric interpretation for the unit imaginary relating it unequivocally to electron spin (Hestenes 1967). I call the reformulation of quantum mechanics in these terms *Real Quantum Mechanics*. Though it has not yet been recognized in the physics mainstream, research on its implications is still underway (Hestenes 2009b).

The third line of research and development was to produce a self-contained system of mathematical tools sufficient for addressing any problem in physics without resorting to alternative mathematical formalisms. The result was a book that defines the domain of *Geometric Calculus* (Hestenes and Sobczyk 1984). The most innovative features of this book are, perhaps, its concepts of *vector manifold, vector derivative and geometric integration theory* (which generalizes Cartan's differential forms).

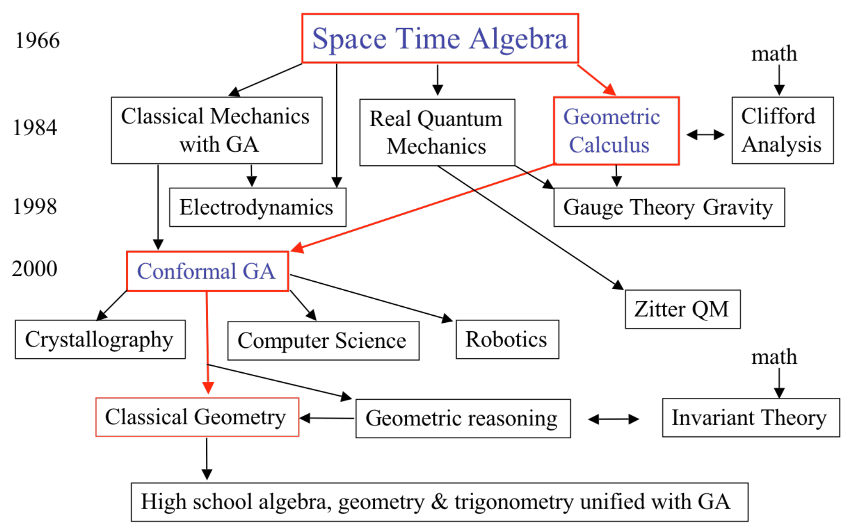

Figure 2. Development of Geometric Algebra and Calculus

Only a handful of people worked with Geometric Algebra until Roy Chisholm (1984) organized an international conference on "Clifford Algebras and their Applications to Physics" that brought together a wide range of mathematicians and physicists with overlapping interests. In particular, it revealed a strong connection between *Geometric Calculus* and an emerging new branch of mathematics called *Clifford Analysis*. The conference (with its published proceedings) was so successful that it has been repeated every four years and is still going strong.

In 1988 a group of theoretical physicists at Cambridge University picked up the threads of Real Quantum Mechanics and Geometric Calculus. The most important result was a new approach to Einstein's General Relativity called *Gauge Theory Gravity* (Lasenby, Doran and Gull 1998). One byproduct was extension of the geometric algebra approach to Lie Groups (Hestenes and Sobczyk 1984) to spin representations for all the classical groups (Doran et.al. 1993). The single most comprehensive treatment of Geometric Algebra for physics is now available in the book (Doran and Lasenby 2003).

A resurgence in applications of Geometric Algebra was ignited when I pointed out the unique advantages of *Conformal Geometric Algebra* for computational geometry (Hestenes 2001). It led directly to more conferences and innovative applications in crystallography (Hestenes and Holt 2007), computer science (Dorst, Fontijne and Mann 2007) and robotics (Bayro-Corrochano and Scheuermann 2009). A compact review of current state of the art is given in (Hestenes 2009a).

Conformal GA appears to be the ideal framework for *Classical Geometry* as envisaged by Felix Klein (1925). We describe some of its surprising new insights below to complete and vindicate Grassmann's approach to geometry. Currently, Conformal GA is at the center of an ambitious research program to master the complexities in advanced *geometric reasoning* (Li 2008). This is intimately related to a branch of mathematics called *invariant theory* (Barnabei, Brini and Rota 1985), and we can anticipate a fruitful interaction if not a merging of these mathematical domains in the future.

Finally, it is worth mentioning that Grassmann's objective to reform the elementary mathematics curriculum is more feasible now than ever before (Hestenes 2002). Steps in this direction are underway, but this is not the place to discuss the complexities of education reform.

Products in Geometric Algebra

Geometric Algebra (GA) today is very close to Grassmann's original algebraic system. To make the connection, we establish the correspondence of current notation, nomenclature and definitions with Grassmann's in his second Ausdehnungslehre (1862). It suffices to consider multiplication. Grassmann's various products were so well conceived that they are still in use today, and, as we shall see, they can all be reduced to the fundamental geometric product in GA.

We begin with an *n-dimensional real vector space* $\mathcal{V}^n = \{a, b, c, \ldots\}$, as defined by Grassmann in perfect accord with the modern concept. As I believe

Grassmann would have it, we use the term "vector" only in this strict algebraic sense. Geometric interpretation will be addressed as a separate matter.

Grassmann's antisymmetric ***outer product*** of two vectors (sometimes called the *join*) is denoted by

$$a \wedge b = -b \wedge a \qquad \leftrightarrow \qquad [ab] = -[ba],$$

with the modern notation on the left corresponding to Grassmann's notation on the right. The *outer product* of k vectors produces *a k-blade* or *k-vector A*:

$$a_1 \wedge a_2 \wedge \ldots \wedge a_k \equiv A \quad \leftrightarrow \quad [a_1 a_2 \ldots a_k] \equiv [A].$$

The integer k is called the *grade* of A.

The outer product of n vectors generates a *pseudoscalar*:

$$a_1 \wedge a_2 \wedge \ldots \wedge a_n = \lambda I \quad \leftrightarrow \quad [a_1 a_2 \ldots a_n] = \lambda,$$

where λ is a scalar and I is the unit pseudoscalar, which has the properties $I \neq 0$, $a \wedge I = 0$ for every vector a. Here we see a subtle design flaw in Grassmann's system, for he identifies pseudoscalars with scalars. This is not a logical mistake, but it complicates the rest of his system (as we see below) and may have kept him from discovering the ultimate simplification with the geometric product.

Grassmann's *regressive product*, sometimes called the ***meet*** (Brini and Teolis 1996, Zaddach 1996, Hestenes and Zeigler 1991) is defined for arbitrary blades A and B in Grassmann's ingenious way by

$$A \vee B = (A^* \wedge B^*)^* \quad \leftrightarrow \quad [A \mid B] = \mid [\mid A \mid B],$$

where the ***dual*** A^* corresponding to Grassmann's *supplement* $|A$ is defined by

$$A^* \equiv \tilde{A} \cdot I = \tilde{A} I \qquad \leftrightarrow \qquad |A,$$

with the ***reverse*** of A defined by

$$\tilde{A} = (a_1 \wedge a_2 \wedge \ldots \wedge a_k)^{\sim} = a_k \wedge \ldots \wedge a_2 \wedge a_1 = (-1)^{k(k-1)/2} A.$$

The dual is often defined without the reverse, which contributes only a sign. The reverse is included here to agree with Grassmann's definition. The definition of dual employs the *inner product* (denoted by a center dot), or better, the *geometric product*, both of which are defined below. Grassmann's *supplement* is based on a concept of orthogonality that amounts to presuming a Euclidean inner product.

For a pair of vectors, Grassmann's *scalar-valued inner product* is expressed by

$$a \cdot b = b \cdot a \quad \leftrightarrow \quad [a \mid b] = [b \mid a].$$

Similarly, Grassmann's regressive product defines a scalar-valued inner product for any pair of blades with the same grade.

Now let us start all over again to define *Geometric Algebra*. Like Grassmann, we introduce an associative (and, of course, distributive) product on the vector space $\mathcal{V}^n = \{a, b, c, \ldots\}$, but now we define it by the simple rule that the square of every vector is a scalar. Explicitly, we write

$$a^2 = aa = \varepsilon\, |a|^2,$$

where scalar $|a| \geq 0$ is called the *magnitude* of *a*, and its *signature* ε is *positive* (ε = 1), *negative* (ε = –1) or *null* (if $|a| = 0$). For any two vectors, we now *define* the ***inner product*** by

$$a \cdot b \equiv \frac{1}{2}(ab + ba) = b \cdot a \,.$$

It is easy to prove that this symmetric product is scalar-valued and is the usual Euclidean inner product if both vectors have positive signature. We also assume that the inner product is *non-degenerate*, which means that every vector has a non-vanishing inner product with some other vector.

Now we define the ***outer product*** by

$$a \wedge b \equiv \frac{1}{2}(ab - ba) = -b \wedge a \,.$$

Adding the last two equations, we see inner and outer products as symmetric and antisymmetric parts of a single ***geometric product***

$$ab = a \cdot b + a \wedge b \,.$$

The definition of the outer product is easily generalized to the antisymmetrized geometric product of any number of vectors to give us *k*-blades, precisely equivalent to those defined above. The inner product can also be generalized to give us

$$aA = a \cdot A + a \wedge A \,.$$

This decomposes the geometric product of a vector with a *k*- blade into a (*k*–1)-blade $a \cdot A$ and a (*k*+1)-blade $a \wedge A$. In other words, the inner product is a grade-lowering operation complementary, or better, dual to the grade-raising outer product. That symmetry is perfectly expressed by the (easily proved) identity

$$a \cdot (AI) = (a \wedge A)I \,.$$

Thus, duality interchanges the roles of inner and outer products.

Now we can formulate the meet as $A \vee B = A \cdot B^*$, the inner product of one blade with the dual of another. This has been used to formulate projective geometry in terms of geometric algebra (Hestenes and Zeigler 1991). That approach is remarkably similar to Grassmann's treatment of projective geometry, but a better approach is described below.

From the vector space $\mathcal{V}^n$ the geometric product generates a ***Geometric Algebra*** $\mathcal{G}^n = \mathcal{G}(\mathcal{V}^n)$. Defining the outer product introduces a *grading* in the algebra that decomposes it into a sum of linear subspaces $\mathcal{G}^n_k$ of homogeneous grade:

$$\mathcal{G}^n = \mathcal{G}(\mathcal{V}^n) = \sum_{k=0}^{n} \mathcal{G}_k^n ,$$

including the scalars $\mathcal{G}_0^n = \mathcal{R}$ as a 1-dimensional subspace. The geometric product also induces a signature (*r,s*) on the vector space that expresses decomposability into an *r*-dimensional subspace of vectors with positive signature and an *s*-dimensional subspace of vectors with negative signature, so that $n = r + s$. This is incorporated in the notation for the algebra by writing $\mathcal{G}^{r,s} = \mathcal{G}(\mathcal{V}^{r,s})$.

Further details about Geometric Algebra are available in publications already mentioned. My purpose here has been to show how perfectly GA incorporates Grassmann's ideas for an *Algebra of Extension*. Indeed, I submit that GA is a next step, perhaps a final step in the evolution of a *universal geometric calculus* first envisaged by Leibniz (Crowe 1967, Grassmann 1995b). Let me summarize the value added by this last step to GA.

In the first place, we have seen that GA reduces the algebraic structure to a single geometric product that mixes grades. Grassmann himself recognized that such a product is needed to incorporate quaternions into his algebraic system, but he did not push his analysis far enough to recognize its fundamental role (Hestenes 1996). Indeed, the geometric product is essential not only to incorporate quaternions (Hestenes 1999), but the whole theory of spinors and spin representations in group theory (Doran et. al. 1993). Below, we see a simple application of the geometric product to congruence.

But what about the strong claim to *universality*!? How can that be justified? Matrix algebra already plays the role of a universal "arithmetic for higher mathematics." Indeed, from its beginning in the middle of the nineteenth century, matrix algebra was a major competitor to Grassmann's system, and it soon swamped his voice. Over the next century matrix algebra was cultivated by legions of mathematicians and physicists to become the dominant mathematical tool in use today. From his Ausdehnungslehre of 1862 it is clear that Grassmann understood the issue deeply (he had already declared his aim to suppress the use of coordinate systems in geometry), but instead of critiquing matrix theory, he set out to show how to handle linear transformations without coordinates. The significance of Grassmann's approach to linear algebra went largely unrecognized, but GA has reinvigorated it with new tools and fully assimilated matrix algebra and all its capabilities (Hestenes 1991, Hestenes and Sobczyk 1984). Without delving into details, it is worth noting that every element of a real matrix can be expressed as the inner product of a pair of vectors; thus

$$a_{ij} = a_i \cdot b_j ,$$

so the inner product plays an essential role in matrix representation. Contrary to common opinion, though, the inner product in GA does not limit its applicability to metric spaces. Rather, it serves the general role of *contraction* (in the sense of that term in tensor algebra). For example, every linear form, that is, every linear mapping φ of vectors into scalars, can be expressed as a contraction; thus

$$\varphi : a \quad \rightarrow \quad \varphi(a) = a \cdot b .$$

Of course, this does not preclude one from the very useful practice of defining a metric tensor by

$$g(a,b) = a \cdot b\,.$$

One last point about matrices: We could "generalize" to complex matrices by introducing complex numbers as scalars. However, that is not advisable, because GA has better (geometrically significant) ways to deal with "complex structure."

As to other algebraic systems, it is well known that every associative algebra has a matrix representation. With matrix algebra incorporated into GA, it follows that every associative algebra can be represented in GA. Even non-associative products can be represented in GA. This has been demonstrated explicitly for the octonian product (Lounesto 1997). If there is an algebraic system that cannot be neatly represented in GA as it stands, then it is likely that GA can be generalized to include it.

Perhaps the ultimate justification for a universality claim is the fact that GA has by far the broadest range of applications to physics and engineering of any single mathematical system (Hestenes 2003), as amply documented by the review in this paper. Furthermore, that review supports an important observation about GA as a language. The discussion of multiplication in the present Section is essentially about defining the *grammar of GA*. But there is far more to a language than its grammar! You need to know how to express important ideas in the language. Thus, creation of *GA as a language* required development of a huge superstructure of definitions, constructions, proofs and calculations to cover classical physics, quantum mechanics. general relativity and engineering applications. Some of it came by insightful translation from other mathematical systems. Some of it involved genuine new insights from exploiting the unique features of GA.

No one has ever been more attuned to the relation of grammar to language than Grassmann. He refined and extracted the geometric structure inherent in synthetic geometry and expressed it in algebraic form. Then he separated the algebra from geometric interpretation to create a general theory of algebraic structures with unlimited dimensions. Besides freeing algebraic structures from the limitations of geometric intuition, Grassmann realized that the same structures can be given many different geometric interpretations. In the following we examine one of the most striking examples of that fact. It involves all the crucial features of GA that have been added to Grassmann's original system: *the geometric product, the pseudoscalar, signature and null vectors.*

Conformal Geometric Algebra

The *Conformal Geometric Algebra* $\mathcal{C}(\mathcal{E}^n)$ for the *n*-dimensional Euclidean space $\mathcal{E}^n$ is defined by $\mathcal{C}(\mathcal{E}^n) \equiv \mathcal{G}(\mathcal{V}^{n+1,1})$. The points of $\mathcal{E}^n$ are represented by null vectors in $\mathcal{V}^{n+1,1}$. Remarkably, the remaining vectors in $\mathcal{V}^{n+1,1}$ represent the hyperplanes and hyperspheres of $\mathcal{E}^n$, and their geometric products generate the entire group of conformal transformations on $\mathcal{E}^n$, including the *Euclidean Group* of rigid displacements and rotations. An example is given in the next Section.

To demonstrate the power and convenience of Conformal GA, we consider the simplest case of the *Euclidean plane* $\mathcal{E}^2$ with its algebra

$$C(\mathcal{E}^2) \equiv \mathcal{G}(\mathcal{V}^{3,1}) .$$

Immediately we encounter an astounding fact: The vector space $\mathcal{V}^{3,1}$ is precisely the standard *Minkowski model* for spacetime, and $\mathcal{G}(\mathcal{V}^{3,1})$ is (except for a trivial difference in sign) precisely the *Spacetime Algebra* that has been so extensively applied to characterize spacetime geometry and physics (Doran and Lasenby 2003, Hestenes 1966).

Thus, the *Conformal Algebra of the Euclidean plane is isomorphic to the Geometric Algebra of Spacetime*! They differ only in geometric interpretation — and a wider difference in interpretation one can hardly imagine! Before describing the geometric interpretation of $C(\mathcal{E}^2)$ in the next Section, let us examine the skeleton of the algebra.

Let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal basis in $\mathcal{V}^{3,1}$. The inner product specifies the signature by:

$$e_0^2 = -1, \quad e_1^2 = e_2^2 = e_3^2 = 1 ,$$

and orthogonality by: $e_\mu \cdot e_\nu = 0$ for $\mu \neq \nu$. The cone of null vectors is depicted in Fig. 3.

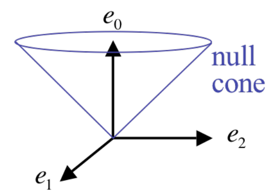

Figure 3.

The outer product generates a basis of six *bivectors*:

$$e_1e_0 ,\quad e_2e_0 ,\quad e_3e_0 ,\quad e_1e_2 ,\quad e_2e_3 ,\quad e_3e_1 ,$$

four *pseudovectors*: $e_0 I ,\ e_1 I ,\ e_2 I ,\ e_3 I ,$

and a unit pseudoscalar: $I = e_0e_1e_2e_3 .$

The ladder of subspaces in $\mathcal{G}(\mathcal{V}^{3,1})$ is depicted in Fig. 4, with dots indicating the dimension of each *k*-vector subspace.

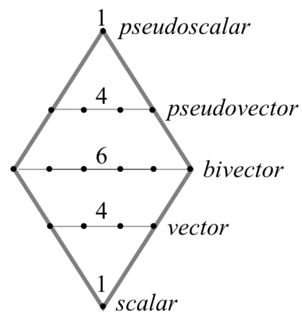

Figure 4.

The Algebra of ruler and compass

As envisioned by Leibniz and conceived by Grassmann, the ultimate goal of geometric calculus is perfect correspondence between algebraic structures, synthetic descriptions in natural language and construction of geometric figures, as summarized by:

Algebraic forms ⇔ Synthetic descriptions ⇔ Geometric figures

Grassmann came close to meeting this goal, but he was roundly criticized by two of his strongest supporters among mathematicians, Friederich Engel and Felix Klein, for an inadequate treatment of projective infinity and ideal (imaginary) figures (Klein 1939). My purpose here is to show how Conformal GA surpasses all expectations in completing Grassmann's program and answering his critics. It is sufficient to consider representations of geometric forms in the algebra of the Euclidean plane. Some elementary calculations will be omitted to concentrate on interpretation of results.

Every *point* p is a null vector, $p^2 = 0$ with *weight* $-p \cdot \infty = 1$, where the symbol ∞ designates the *point at infinity*, with $\infty^2 = 0$. The inner product determines the distance d_{21} between two points:

$$d_{21}^2 = (p_2 - p_1)^2 = -2 p_2 \cdot p_1 \geq 0 ,$$

which vanishes if the points coincide, thus justifying the representation of points by null vectors. Grassmann himself made an intuitive distinction between points and the difference between points like $p_2 - p_1$ (Grassmann 1995b). Now the distinction is encoded as an algebraic difference between two kinds of vectors, null and non-null. We shall see that this distinction has some surprising new implications. Before continuing, though, it may be worth confirming the familiar properties of a Euclidean triangle determined by points p_1, p_2, p_3. The sides of the triangle can be represented by vectors $p_{ij} = p_i - p_j$, which implies the triangle equation $p_{21} + p_{32} + p_{13} = 0$, whence the familiar *law of cosines*

$$p_{21}^2 + p_{32}^2 + 2 p_{21} \cdot p_{32} = p_{13}^2 .$$

This can be regarded as an implicit definition for the cosine of the included angle, where $p_{12} \cdot p_{32} = d_{12} d_{32} \cos\theta_{13}$. Less familiar is the fact that the trigonmetric *law of sines* is expressed by

$$p_{21} \wedge p_{32} = p_{32} \wedge p_{13} = p_{13} \wedge p_{21} ,$$

with *directed area A* for the triangle given by

$$\begin{aligned} 2A \equiv p_{12} \wedge p_{32} &= (p_1 - p_2) \wedge (p_3 - p_2) \\ &= p_1 \wedge p_3 - p_1 \wedge p_2 + p_3 \wedge p_2 = (p_1 \wedge p_2 \wedge p_3) \cdot \infty, \end{aligned}$$

and magnitude specified by

$$4|A|^2 = (p_{12} \wedge p_{32}) \cdot (p_{32} \wedge p_{12}) = p_{32}^2 p_{12}^2 - (p_{12} \cdot p_{32})^2 = d_{32}^2 d_{12}^2 \sin^2\theta_{13} .$$

Now things get more interesting.

A *circle S* is generated by the product of three points: $S = p_1 \wedge p_2 \wedge p_3$. This algebraic form can be interpreted as an instruction for drawing a circle through three given points, as depicted in Fig. 5. Each *line L* is a circle through ∞, as expressed by $L = p_1 \wedge p_2 \wedge \infty$ and also depicted in Fig. 5. Circles and lines are *oriented*, as expressed by a change in sign induced by interchanging the order of points; thus, $-S = p_3 \wedge p_2 \wedge p_1$ and $-L = p_2 \wedge p_1 \wedge \infty$. An orientation for the circle, but not the line, is indicated in Fig. 5. In many applications, such as projective geometry, the orientation is not of interest, so the sign can be ignored.

It is often more convenient to represent geometric objects by their duals. The *dual forms* for both lines and circles are non-null vectors. In particular, the dual of line L is the *normal* for the line $l = L^* = LI$. The normal for a line is distinguished from other vectors by being orthogonal to the point at infinity; that is,

$$L \wedge \infty = 0 \quad \Leftrightarrow \quad l \cdot \infty = 0 .$$

This property implies that the normal can be expressed as the difference between two points: $l = q_2 - q_1$, as depicted in Fig. 6. Conversely, the vector difference between any two points is the "perpendicular bisector" of the line through those points. The line has a magnitude given by

$$L^2 = \left(p_2 - p_1\right)^2 = l^2 = \left(q_2 - q_1\right)^2 > 0 ,$$

where the points are those depicted in Figs. 5 & 6. Consequently, L can be interpreted as a line segment, though it does not specify a location of the segment along the line.

The dual of circle S is a vector

$$S^* = s = c - \frac{1}{2}\rho^2\infty ,$$

which, like the normal for a line, is the difference between points; but, in this case, one point is the center of the circle c, while the other is the weighted point at infinity. Indeed, it is readily verified that $c^2 = 0$ and $s^2 = \rho^2$. To verify that scalar ρ is the radius of the circle, note that the condition for a point p to lie on the circle is

$$p \wedge S = 0 \quad \Leftrightarrow \quad p \cdot s = 0 .$$

The usual equation for a circle then follows easily:

$$\left(p - c\right)^2 = -2p \cdot c = \rho^2 .$$

Thus we have discovered the surprising fact that the specification of a circle by its center and radius (in Fig. 7) is dual to its specification by three points (in Fig. 5).

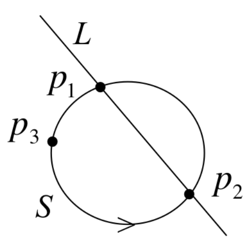

Figure 5.

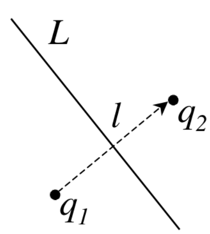

Figure 6.

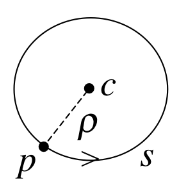

Figure 7.

One advantage of Conformal GA is that projective geometry is fully integrated with metrical geometry. For example, the *incidence relations* among points and lines are expressed directly by the meet product. Thus, the incidence (intersection) of a point p with line L is expressed by

$$p \vee L = p \cdot L^{*} = p \cdot l = 0\ ,$$

as depicted in Fig. 8. Parallelism of lines depicted in Fig. 9 is expressed by

$$L_1 \vee L_2 = 0 \quad \Leftrightarrow \quad l_1 \wedge l_2 = 0\ .$$

The intersection of lines at a point depicted in Fig. 10 is expressed by

$$p \vee (L_1 \vee L_2) = p \cdot (l_1 \wedge l_2) = 0\ .$$

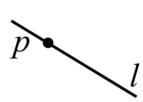

Figure 8.

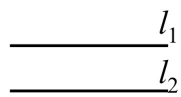

Figure 9.

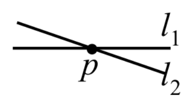

Figure 10.

The projective incidence relations for intersecting objects generalize automatically to metrical relations for objects in arbitrary positions. This could be called *metric incidence*. Thus, as depicted in Fig. 11, the distance δ between a point and a line is given by

$$p \cdot \hat{l} = \pm\delta, \quad \hat{l}^2 = 1\ ,$$

where the sign depends on the orientation of the line, so it distinguishes on which side of the line the point lies.

As depicted in Fig. 12, the distance between a point and a circle is given by

$$s \cdot p = c \cdot p + \frac{1}{2}\rho^2 = \frac{1}{2}\left[\rho^2 - d^2\right],$$

where d is the distance from the point to the center of the circle. Clearly, the sign of $s \cdot p$ specifies whether the point is inside or outside the circle. It is therefore a topological property expressing relative orientation. More generally, if signs are retained but magnitudes are ignored, we have a generalization of projective geometry to include orientation that is very useful in geometric computation (Stolfi 1991).

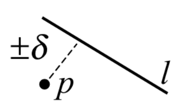

Figure 11.

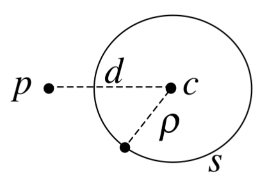

Figure 12.

Since a point can be regarded as a circle with zero radius, we have the obvious generalization to a distance between two circles:

$$s \cdot s' = (c - \frac{1}{2}\rho^2\infty) \cdot (c' - \frac{1}{2}\rho'^2\infty) = \frac{1}{2}(\rho^2 + \rho'^2 - d^2) .$$

Now we are prepared to give a response to the critique of Engel and Klein that Grassmann's algebra cannot handle *ideal geometric forms*. The sum of two points gives us something that we not seen before:

$$\overline{s} \equiv \frac{1}{2}(p_1 + p_2) = c + \frac{1}{2}\rho^2\infty .$$

This can be regarded as an *imaginary circle*, because $\overline{s}^2 = -\rho^2$. Imaginary circles arose first as complex solutions of quadratic equations, and then demanded a geometric interpretation (Klein 1939). We have here a new possibility.

It is an old idea that *lines can be regarded as points at infinity*. Accordingly, we can represent lines by their normals, and define a *circle at infinity* as the join of two normals with the circle *s* defined before:

$$\overline{S} = l_1 \wedge l_2 \wedge s = (l_1 \wedge l_2)s .$$

Its dual form is the vector $\overline{s} = \overline{S}^*$. Since $\overline{S}^2 = (l_1 \wedge l_2)^2 s^2$, we can normalize with $(l_1 \wedge l_2)^2 = -1$, so $\overline{s}$ has the same *center c* and *radius* ρ as the *real circle s*.

Though the circle center is a *real point* (a null vector), there can be no real point p on the imaginary circle, because the equation $\overline{s} \cdot p = c \cdot p - \rho^2 / 2 = 0$ has no null vector solutions. However, the equation $\overline{s} \cdot l = c \cdot l = 0$ does have imaginary points as its solution. Indeed, the solution set consists of all lines with weight ρ passing through the circle center. Thus, we see that *an imaginary circle is just a representation of a real circle by the family of lines through its center.*

As far as I know, this is a completely new perspective on imaginary circles, and it illustrates how Conformal GA can provide spectacular rejoinders to objections by Grassmann's critics.

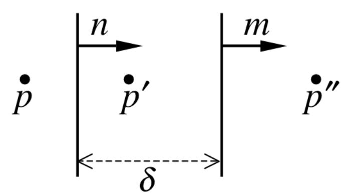

Figure 13.

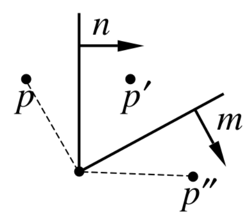

Figure 14.

All the geometric constructions so far have involved only inner and outer products. It remains to demonstrate advantages of using the geometric product directly. Let n be the unit normal for a given line. Then it is easy to prove (Hestenes 2002) that the *reflection* of every point p (hence every geometric object) across that line is specified by the transformation

$$p \quad \mapsto \quad p' = -npn \,,$$

as shown in Fig. 13. Following this with a second reflection across a line with unit normal m, the net result is a *rigid displacement*:

$$p \quad \mapsto \quad p'' = Dp\tilde{D}$$

where $D = mn$ with reverse $\tilde{D} = nm$. If the two lines are parallel, the displacement is a *translation* through twice the distance δ between the lines (Fig. 13). If the lines intersect at a point, the displacement is a *rotation* about that point through twice the angle between the lines (Fig. 14). Thus, the well-known synthetic description of translations and rotations in terms of reflections through lines (or planes in $\mathcal{E}^3$) has been reduced to the simple geometric product of vectors. One consequence is considerable simplification in the treatment of crystallographic symmetries (Hestenes and Holt 2007).

The present formulation for rigid displacements, hence of *congruence*, applies without change in form to symmetries in $\mathcal{E}^n$. In $\mathcal{E}^3$ it provides the foundation for powerful engineering applications. For example, in rigid body dynamics it unifies translational and rotational equations (Newton's and Eulers Laws) into a single equation of motion (Hestenes 2009a).

I believe Grassmann would be greatly pleased!!

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