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Conceptual Modeling
in physics, mathematics and cognitive science

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Abstract: Scientific thinking is grounded in the evolved human ability to freely create and manipulate *mental models* in the imagination. This *modeling* ability enabled early humans to navigate the natural world and cope with challenges to survival. Then it drove the design and use of *tools to shape and control* the environment. Spoken language facilitated the sharing of mental models in cooperative activities like hunting and in maintaining tribal memory through storytelling. The evolution of culture accelerated with the invention of written language, which enabled creation of powerful symbolic systems and *tools to think with*. That includes deliberate design of mathematical tools that are essential for physics and engineering . A mental model coordinated with a symbolic representation is called a *conceptual model.* Conceptual models provide symbolic expressions with meaning.

This essay proposes a *Modeling Theory* of cognitive structure and process. Basic definitions, principles and conclusions are offered. Supporting evidence from the various cognitive sciences is sampled. The theory provides the foundation for a science pedagogy called *Modeling Instruction*, which has been widely applied with documented success and recognized most recently with an *Excellence in Physics Education* award from the American Physical Society.

The *Copernican Revolution* in science culminated in Newton‘s *Principia* (1687), which integrated astronomy and terrestrial physics into a single science of motion. Immanuel Kant (1787) saw this as a striking union of mathematical theory with empirical fact that bridged the traditional divide between rationalism and empiricism. So he proposed a comparable *"Copernican Revolution"* in philosophy to account for it [1]. Just as Copernicus shifted the center of the universe from earth to sun, Kant shifted the focus of epistemology from structure of the external world to structure of mind. His revolutionary insight was that our perceptions and thoughts are shaped by inherent structure of our minds. He argued that the fundamental laws of nature, like the truths of mathematics, are knowable precisely because they do not describe the world as it really is but rather prescribe the structure of the world as we experience it.

Though the scientific revolution has expanded in spectacular fashion to integrate physics and astronomy with chemistry and biology, Kant's revolution in philosophy has hardly progressed. His profound influence on the epistemology of physics is evident in the writings of Einstein and Bohr as well as many other scientists and philosophers. However, continued debates on such topics as the interpretation of quantum mechanics show no signs of consensus, and they have overlooked recent advances in cognitive science with high relevance to epistemology.

My purpose here is to open a new stage in Kant's revolution by explaining how findings of cognitive science can be marshaled to create a new "*science of mind*" with testable predictions and explanations as required of any "true" science. I begin with a restatement of Kant's primary question: *What does the structure of science and mathematics tell us about how the human mind works*? In searching for answers my working hypothesis will be: *The primary cognitive activities in science and mathematics involve making, validating and applying conceptual models!* In a word, science and mathematics are about MODELING — making and using models!

This essay argues for a "MODELING THEORY of MIND" to guide the multifarious branches of cognitive science in research on the nature of mind and brain, and the design of conceptual tools for science and mathematics. Core principles are explained and supporting evidence is sketched, but the brush is necessarily broad. More details are given in [2,3,4], especially for application to physics teaching and learning.

I. NEWTON'S MODELING GAME

Newton did much more than provide the first mathematical formulation of a scientific theory in his *Principia*; he also demonstrated how to relate it to empirical fact. Though Kant recognized revolutionary implications for epistemology in this impressive feat, physicists have overlooked it. The issue has been thoroughly explicated in [5] by framing Newtonian theory in terms of models and modeling, so brief mention of key points is sufficient here. Newton could not make the crucial distinction between model and theory explicit in his original formulation, because the concept of model did not emerge in scientific discourse until the nineteenth century. But [5] shows that he made it implicitly. The point is that theoretical principles like Newton's Laws cannot be tested or applied except by incorporating them in models. Thus, models mediate between theory and experiment. And Newton's Laws can be regarded as a system of design principles for making models to describe, to predict, to explain and to control motions of material bodies.

Kant's insight can be explicated by noting that Newton linked up two distinct kinds of models: theoretical and empirical. A *theoretical model* derived from Newton's Laws predicts motions, while an *empirical model* derived from data describes a motion. A match between them explains a motion. In this way Newton explained Galileo's law of falling bodies and Kepler's three laws of planetary motion. Note the distinction between a theoretical <u>Law</u> (with a capital L) and an empirical <u>law</u> (with a lowercase l), also called an empirical model.

Comparison between theoretical and empirical models is such a standard practice of physicists since Newton that they seldom consider its profound epistemological implications. At its simplest, it involves creating an empirical model from data with a procedure often called "curve fitting." That's how Kepler's laws were derived. It is an important technique in the search for empirical regularities that are both quantifiable and reproducible. In high energy physics data analysis has become so complex that a new research specialty has emerged to handle it. That research, often called "phenomenology," is thus intermediate between theory and experiment.

For future analysis, it is worth noting that scientific work in all three domains is governed by definite but different rules; from mathematical rules for theorists, to measurement standards for experimentalists, to probability theory for phenomenologists. As Kant recognized, scientific objectivity requires strict adherence to rules. The question is: Where do the rules come from?

II. FROM COMMON SENSE TO SCIENTIFIC THINKING

As we grow and learn through everyday experience, each of us develops a system of *common sense* (CS) concepts about how the world works. To evaluate introductory physics instruction, the *Force Concept Inventory* (FCI) was developed to detect differences (in student thinking) between CS concepts and Newtonian concepts about motion and its causes [6]. Results from applying the FCI were stunning from the get-go! First, the differences were huge before instruction. Second, the change was small after instruction. Third, results were independent of the instructor's experience, teaching method and peer evaluation. These results have been replicated thousands of times from high school to Harvard and in 25 different languages. The FCI is by far the most cited reference in the physics education literature, and it is widely used today to evaluate the effectiveness of teaching reforms.

Here we are interested in what the FCI tells us about human cognition. The FCI is based on a taxonomy of 35 CS concepts in 5 major categories [6]. These concepts are overlooked or summarily dismissed as misconceptions by most physicists. However, they are common outcomes from everyday experience, and they are quite serviceable for dealing with physical objects. Moreover, central CS concepts in the 5 categories have been clearly articulated and discussed by major intellects of the pre-Newtonian age, including Newton himself before the *Principia* [7]. So CS concepts should be regarded as alternative hypotheses about the physical world that, when clearly formulated, can be tested empirically.

For example, the CS concept: "a moving object implies existence of a force (a mover)" contravenes Newton's First Law. The Second Law is contravened by the concept that forces are due to "active agents" (usually living things), so there are no passive forces, although motion is deflected by passive objects called "barriers." The Third Law is contravened by the common metaphorical notion that "interaction is like war" so in the "struggle between forces" "victory goes to the stronger." In fact, CS thinking is shot through with metaphorical notions. One consequence of all this is that in a conventional physics course students systematically misinterpret what they hear and see in class, which goes a long way to account for the typical disastrous student performance on exams.

Ability to distinguish between CS concepts and scientific concepts in the FCI or elsewhere is not a matter of intelligence but of experience. It is acquired only by engagement with science itself, usually through academics. Remarkably, physicists seldom recall any event in their own transitions to Newtonian thinking. Typically, they presume that the world of classical physics is given directly by experience, in contrast to the subtlety and weirdness of quantum mechanics. They are blind to the subtle revolution in their own thinking that came from learning physics; for the FCI tells us that classical physics differs from common sense in almost every detail.

These facts suggest that the transition from common sense to scientific thinking is not a replacement of CS concepts with scientific concepts, but rather a realignment of intuition with experience. Science does not replace common sense. Rather, as Modeling Theory aims to show, science is a refinement of common sense differing in respect to:

- <u>objectivity</u> – with explicit rules & conventions for observer-independent inferences,
- <u>precision</u> – in measurement, in description and analysis,
- <u>formalization</u> – for mathematical modeling and analysis of complex systems,
- <u>systematics</u> – coherent, consistent & maximally integrated bodies of knowledge,
- <u>reliability</u> – critically tested & reproducible results,
- <u>skepticism</u> – about unsubstantiated claims.

III. "What, precisely, is thinking?" — *Einstein*

Kant is unsurpassed in using introspection to analyze his own thinking. But introspection was dismissed as subjective and unreliable by behaviorists in the twentieth century, who claimed that scientific objectivity requires psychology to take its data from observable behavior under controlled conditions. However, the behaviorist straight jacket has been cast off in recent decades by the emergence of *cognitive science*, which draws its data and insights from many independent academic disciplines. Disciplinary barriers are crossed with increasing frequency, largely due to the speed and ease of electronic communication.

Human perception, memory and cognition are being studied in many different ways. The problem, as ever in science, is to identify reproducible patterns in the results. Here follows a sampling of approaches and results with high relevance to Modeling Theory.

<u>The Learning Sciences</u>: Research on teaching and learning is emerging as a coherent science with independent branches like physics education research (PER) devoted to a single discipline. The most robust finding in the field is that effective teaching requires matching the method to the subject matter, and that requires research embedded within each discipline.

An outstanding example of PER is Andy diSessa's probing study into common sense notions of force [8]. He identifies a structure in common sense intuition that he calls Ohm's p-prim. As he explains,

> **Ohm's p-prim** comprises "an <u>agent</u> that is the locus of an <u>impetus</u> that acts against a <u>resistance</u> to produce a <u>result</u>."

Evidently this intuitive structure is abstracted from experience in pushing objects. It is an important characterization of the central Force-as-Action metaphor identified by the FCI. It also seems to be fundamental in the intuition of physicists, who often declare "A force is a push or a pull," although they have extracted that from human action.

More generally, diSessa argues that this structure is fundamental to qualitative reasoning. He notes that the logic of Ohm's p-prim is

the *qualitative proportion*: more effort ⇒ more result,
and the *inverse proportion*: more resistance ⇒ less result.

This reasoning structure is often evoked for explanatory purposes in everyday experience.

As disclosed in Ohm's p-prim, the concept of (causal) **agency** entails a basic

Causal syntax: agent $\rightarrow$ (kind of action) $\rightarrow$ on patient $\rightarrow$ result.

DiSessa notes that this provides an interpretative framework for $\mathbf{F} = m\mathbf{a}$, and he recommends exploiting it in teaching mechanics. However he does not recognize it as a basic aspectual schema for verb structure, which has been studied at length in *cognitive grammar* [9]. Aspectual concepts are generally about event structure, where events are changes of state and causes (or causal agents) induce events.

All this has direct bearing on Kant's *Critique*. He said Hume woke him from his "dogmatic slumber" with his argument that no amount of empirical data can establish a cause-effect relation between events with certainty. Claiming that Newton's Laws do establish causality with certainty, Kant argued that it must therefore be known prior to experience ("synthetic a priori"). One can argue instead that the fundamental Laws and Principles of science are discovered as general patterns in experience and simply adopted as postulates in our theories. But Ohm's p-prim shows that causality is imbedded in the way we think and so it may be a precondition for recognizing causal patterns. In this sense, at least, cognitive science supports Kant's view.

All this has bearing on other domains of cognitive science, for example, the psychology of perception. In particular, it provides more support for the view that there is no such thing as "passive perception." All perception is part of a "**perception-action cycle**." Even viewing a static visual scene is impossible without rapid movements of the eyeballs (saccades) to sample the visual input. In general, we learn about the world around us from our interactions (perception-actions) with it.

Note that the intuitive causal syntax discussed above can be construed (by metaphorical projection at least) as

Operator syntax: agent $\rightarrow$ (kind of action) $\rightarrow$ on patient $\rightarrow$ result,

where the action is on symbols (instead of material objects) to produce other symbols. When the symbols are words, this provides an intuitive base for verb structure expressing the action of mental agents on mental objects. The same idea has emerged independently in cognitive linguistics (see below). Also, the operator syntax provides an intuitive base for the mathematical concept of function (though probably not the only one).

<u>Narrative Comprehension</u>: Readers of stories construct mental models of the situation and characters described [10]. They infer causal connections relating characters' actions to their goals. They also focus attention on characters' movements, thereby activating nearby parts of the mental model. This activation is revealed in readers' faster answering of questions about such parts, with less facilitation the greater their distance from the focus. Recently visited as well as imagined locations are also activated for several seconds. These patterns of temporary activation facilitate comprehension.

<u>Evolutionary Psychology</u> [11] tells us that human brains evolved adaptively to enable navigation to find food and respond to threats. Successful hunting required a number of cognitive abilities: To create mental maps of the environment and plan actions, to design helpful tools, to "read" subtle clues in natural surroundings; and, finally to communicate and cooperate with other humans.

There can be little doubt that narrative emerged in human prehistory. The practice of storytelling is ancient, pre-dating not only the advent of writing, but of agriculture and permanent settlement as well. Language, an obvious prerequisite for storytelling, is likely to have emerged between 50,000 and 250,000 years ago. Cognitive linguistics (see below) aims to ascertain what language can tell us about evolved cognitive abilities.

<u>Cognitive Psychology</u>: Psychologist Philip Johnson-Laird [12] is a pioneer in studying human inference by manipulating mental models. His research supports the claim that *most human reasoning is inference from mental models*. We can distinguish several types of **model-based reasoning**:

- **Abductive,** to complete or extend a model, often guided by a semantic frame in which the model is embedded.
- **Deductive**, to extract substructure from a model.

- **Inductive**, to match models to experience.
- **Analogical**, to interpret or compare models.
- **Metaphorical***,* to infuse structure into a model.
- **Synthesis,** to construct a model, perhaps by analogy or blending other models.
- **Analysis,** to profile or elaborate implicit structure in a model.

Justification of model-based reasoning requires translation from mental models to *inference from conceptual models* that can be publicly shared, like the scientific models discussed below.

In contrast, **formal reasoning** is computational, using axioms, production rules and other procedures. It is the foundation for rigorous proof in mathematics and formal logic. However, Modeling theory (see below) holds that mathematicians and even logicians reason mostly from mental models. Model-based reasoning is more general and powerful than propositional logic, as it integrates multiple representations of information (propositions, maps, diagrams, equations) into a coherently structured mental model. Rules and procedures are central to the formal concept of inference, but they can be understood as prescriptions for operations on mental models as well as on symbols.

Psychology of Spatial Perception: Everyone has imagination, the ability to conjure up an image of a situation from a description or memory. What can that tell us about mental models? Some people report images that are picture-like, similar to actual visual images. However, others deny such experience, and blind people are perfectly capable of imagination. Classical research in this domain found support for the view that *mental imagery is internalized perception,* but not without critics.

Barbara Tversky and collaborators [13] have tested the classical view by comparison to mental model alternatives. Among other things, they compared individual accounts of a visual scene generated from narrative with accounts generated from direct observation and found that they are *functionally equivalent.* A crucial difference is that perceptions have a fixed point of view, while mental models allow change in point of view. Furthermore, spatial mental models are more schematic and categorical than images, capturing some features of the object but not all and incorporating information about the world that is not purely perceptual. The general conclusion is that *mental models represent states of the world as conceived, not perceived*. To know a thing is to form a mental model of it.

Major characteristics of spatial mental models are summarized in the following list. The best fit to data is a *spatial framework model*, where each object has an *egocentric frame* consisting of mental extensions with three body axes.

Spatial MENTAL models

- are ***schematic***, representing only some features,
- are ***structured,*** consisting of *elements and relations*.
- **Elements are typically *objects*** (or reified things).
- **Object *properties* are idealized** (points, lines or paths).
- Object models are always ***placed in a background*** (context or **frame**).
- Individual objects are ***modeled separately*** from the frame,
  so they can move around in the frame.

The details in this list are abundantly supported by other lines of research, especially in cognitive linguistics, to which we now turn.

Cognitive Linguistics: The most extensive and coherent body of evidence comes from cognitive linguistics [14], supporting the **revolutionary thesis**: *Language does not refer directly to the world, but rather to mental models and components thereof! Words serve to activate, elaborate or modify mental models, as in comprehension of a narrative.*

This thesis rejects all previous versions of semantics, which located the referents of language outside the mind, in favor of **cognitive semantics**, which locates referents inside the mind. I see the evidence supporting cognitive semantics as overwhelming, but it must be admitted that some linguists are not convinced, and many research questions remain. Cognitive semantics can be regarded as a culmination of Kant's revolution toward an epistemology grounded in science, though that is not often recognized by linguists.

Two pillars of cognitive linguistics deserve mention here. The first pillar is Eleanor Rosch's discovery that ***natural categories*** are determined by mental prototypes. For example, "birds" are classified by comparison to a prototypical bird, such as a robin. This should be contrasted with the classical concept of a ***formal category*** for which membership is determined by a set of defining properties, a noteworthy generalization of the container metaphor. This distinction between category types is supported by a mountain of empirical evidence on natural language use.

The second pillar is the notion of ***image schema*** introduced by Mark Johnson and George Lakoff. Image schemas are basic *structural units* (gestalts) that provide structure to natural language and presumably cognition. There are too many to discuss here. Many are discussed in [15] as structural elements in mathematical thinking, including four grounding metaphors for arithmetic.

<u>Cognitive Neuroscience</u>: Human brain structures have evolved to support perception, memory and movement, that is, all components needed to execute the perception-action cycle. But no distinct component for cognition has been identified. It seems reasonable, therefore, to conclude that cognition is executed by coopting drivers of the perception-action for internal planning and simulation.

Stanislas Dehaene reports [16]: "Mathematicians frequently evoke their "intuition" when they are able to quickly and automatically solve a problem, with little introspection into their own insight. Cognitive neuroscience research shows that "automaticity aspect" of mathematical intuition can be studied in the laboratory in reduced paradigms, and that relates to the availability of "core knowledge" associated with evolutionarily ancient and specialized cerebral subsystems." Subsystems involved in basic operations of arithmetic (such as number estimation, comparison, addition and subtraction) have been identified as genetically hardwired. The boundary between hardwired and learned mathematical abilities continues to be a rich area for further research.

The empirical research cited above supports an answer to Einstein's question: *Thinking* is a hardwired human ability to freely create mental models and use them for planning and controlling interactions with the physical world. To deepen this insight and coordinate empirical results, we need a scientific theory, to which we now turn.

IV. MODELING THEORY

Though Modeling Theory is proposed as a general theory of Mind embracing all aspects of cognitive science, we limit our attention here to cognition in physics and mathematics. We have seen above that the study of natural languages gives us rich information about the structure of mental models in common sense cognition. Given its greater precision and coherence, we can expect complimentary and reinforcing results from studying the language of science, especially mathematics. Indeed, after spelling out the structure of **scientific models** in explicit detail below, we discuss its implications for cognition in physics and mathematics.

Our formulation of Modeling Theory rests on explication of two key concepts "model" and "morphism." We begin with the definition:

A **model** is a **<u>representation of structure</u>** in a given **system**.

A *system* is a set of related *objects*, which may be real or imaginary, physical or mental, simple or composite. The *structure* of a system is a set of relations among its objects. The system itself is called the *referent* of the model.

We often identify the model with its *representation* in a concrete inscription of words, symbols or figures (such as graphs, diagrams or sketches). But it must not be forgotten that the inscription is supplemented by a system of (mostly tacit) rules and conventions for encoding model structure.

From my experience as a scientist, I have concluded that **five types of structure suffice** to characterize *any scientific model*. Although my initial analysis was based on physics, I have concluded the classification is sufficient for all other sciences as well. As this seems to be an important empirical fact, a brief description of each type is in order here.

Universal structures in scientific models [2, 17]:

- **Systemic structure:** Its representation specifies (a) *composition* of the system (b) *links* among the parts (individual objects), (c) links to *external agents* (objects in the environment). A diagrammatic representation is usually best (with objects represented by nodes and links represented by connecting lines) because it provides a wholistic image of the entire structure. Examples: electric circuit diagrams, organization charts, family trees.

- **Geometric structure:** specifies (a) *configuration* (geometric relations among the parts), (b) *location* (position with respect to a reference frame)

- **Object structure:** *intrinsic properties* of the parts. For example, mass and charge if the objects are material things, or *roles* if the objects are *agents* with complex behaviors. The objects may themselves be systems (such as atoms composed of electrons and nuclei), but their internal structure is not represented in the model, though it may be reflected in the attributed properties.

- **Interaction structure:** properties of the links (typically *causal* interactions). Usually represented as binary relations on object pairs. Examples of interactions: forces (momentum exchange), transport of materials in any form, information exchange.

- **Temporal (event) structure:** *temporal change in the state* of the system. Change in position (motion) is the most fundamental kind of change, as it provides the basic measure of time. Measurement theory specifies how to quantify the properties of a system into property variables. The state of a system is a set of values for its property variables (at a given time). Temporal change can be represented *descriptively* (as in graphs), or *dynamically* (by equations of motion or conservation laws).

Optimal precision in definition and analysis of structure is supplied by **mathematics, the science of structure.**

Both the model and its referent are structured objects, but they need not be distinct. Indeed, the usual notion of a *mathematical model* as a representation in terms of mathematical symbols does not specify any referent, so we say it is an *abstract model.* Of course*,* it is a perfect representation of itself. This suggests that we regard any structured object as an "abstract model."

Our definition of "model" above is likewise abstract, because it does not specify the worlds (domains) in which the model and its referent exist as structured objects. To address this issue, Modeling Theory [3] posits three distinct worlds in which structured objects exist:

- World 1: The PHYSICAL WORLD of real things and events, including biological entities.
- World 2: The MENTAL WORLD of **mental models** generated by perception or intuition.
- World 3: The CULTURAL WORLD of human artifacts, including natural languages and mathematics in any form, written or spoken.

This helps us make a crucial distinction between mental models and conceptual models**. Mental models** are private constructions in the mind of an individual (World 2). They can be elevated to **conceptual models** by encoding model structure in symbols (World 3) that activate the individual's mental model and corresponding mental models in other minds. Thus, communication between individuals involves construction and use of shared *conceptual models*.

Note that a conceptual model establishes an *analogy* between a mental model and its symbolic representation. Mathematical models are symbolic structures, and to understand one is to create a mental model with analogous structure. Actually, *the structure is supplied by the mind not the symbols*, which are reduced to meaningless marks without a mind to interpret them.

An ***analogy*** is defined as a *mapping of structure* from one domain (*source*) to another (*target*) [18]. The mapping is always partial, which means that some structure is not mapped. Science sets up many kinds of analogy between and within the three worlds [3]. Thus, experimental testing or simply interpreting a scientific model (World 3) requires a mapping to a physical system (World 1) that I call a *referential analogy. Material analogies* relate structures of different physical objects in World 1 and this reduces to an *inductive analogy* when the objects are regarded as identical. And there are many more analogies with *computer models* (World 1).

There are other kinds of structure-preserving mappings such as *metaphors*, which Lakoff [15] defines as a projection of structure from one domain into another. I recommend formalizing all such concepts with the technical term **MORPHISM**. In mathematics a ***morphism*** *is a structure-preserving mapping*: Thus the terms *homomorphism* (preserves algebraic structure) and *homeomorphism* (preserves topological structure).

Now let us reconsider Kant's trenchant analysis of thinking in physics and mathematics. *Physical intuition* is accorded the same high regard by physicists that mathematicians accord to *mathematical intuition.* To quote unquestionable leaders in each field [2]:
Einstein explains,

> "The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. . . . The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be voluntarily reproduced and combined. . . . "

Hilbert asserts,

> "No more than any other science can mathematics be founded on logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought."

Modeling theory asserts that physical and mathematical intuitions are merely two different ways to relate products of imagination to the external world. **Physical intuition** matches structure in mental models with structure in physical systems. **Mathematical intuition** matches mental structure with symbolic structure. Thus, structure in imagination is common ground for both physical and mathematical intuition.

Kant reasoned in much the same way. He also took the physics and mathematics of his day as given and asked what makes them so special. His analysis is cogent even today, so key points are worth reconsidering. He began by identifying ***construction in intuition*** as ***a means*** for acquiring certain geometrical knowledge:

> "Thus we think of a triangle as an **object**, in that we are conscious of the combination of the straight lines according to a **rule** by which such an **intuition** can always be **represented**. . . This representation of a universal procedure of imagination in providing an image for a concept, I entitle the **schema of this concept**."

Kant did not stop there. Like any good scientist he anticipated objections to his hypothesis. Specifically, he noted that his intuitive image of a triangle is always a *particular triangle.* How, he asks, can construction of a concept by means of a single figure "express universal validity for all possible intuitions which fall under the same concept?" This is the general epistemological *problem of universality* for the case of Kant's theory of geometrical proof. Kant's notion of geometrical proof is by construction of figures, and he argues that such proofs have universal validity as long as the figures are "determined by certain universal conditions of construction." In other words, construction in intuition is a ***rule-governed activity*** that makes it possible for geometry to discern "the universal in the particular."

Kant's argument is often dismissed because it led him to conclude that Euclidean geometry is *certain a priori*. But that is a red herring! Because we now know that non-Euclidean geometry can be associated with the same intuitive construction simply by changing the rules assigned to it. His essential point is that **mathematical inference from intuition is governed by subsumption under rules**. As mathematician Saunders MacLane [19] asserts, "Mathematics is not concerned with reality but with rule."

V. RULES AND TOOLS FOR THINKING AND DOING

Science and technology have coevolved with language and mathematics. The evolution is driven by invention of tools with increasing sophistication and power to shape and understand the physical world. The tools of science are of two kinds: instruments for detecting reproducible patterns in the material world, and symbolic systems to represent those patterns for contemplation in the mind.

The detection of patterns in nature began with direct observation using human sensory apparatus. Then the human perceptual range was extended by scientific instruments, such as telescopes and microscopes. Finally, Technology has replaced human sensory detectors with more sensitive instruments, and the data is processed by computers with no role for humans except to interpret the final results; even there the results may be fed to a robot to take action with no human participation at all.

Tool development in the cognitive domain began with the natural languages in spoken and then written form. Considering their *ad hoc* evolution, the coherence, flexibility and subtlety of the natural languages is truly astounding. More deliberate and systematic development of symbolic tools came with the emergence of science and mathematics. The next stage of enhancing human cognitive powers with computer tools is just beginning.

While science is a search for structure, mathematics is the science of structure. Every science develops specialized modeling tools to represent the structure it investigates. Witness the rich system of diagrams that chemists have developed to characterize atomic and molecular structure. Ultimately, though, these diagrams provide grist for mathematical models of greater explanatory power. What accounts for the ubiquitous applicability of mathematics to science? An answer is suggested by considering the coevolution of mathematics and physics from the perspective of modeling theory.

Tools of technology provide an obvious index of progress in human civilization, because their results are so tangible. A more subtle and informative index is the development of language and mathematics, which provide us with **tools to think with**! Though spoken language reaches back more than 150,000 years, written language is barely 5,000 years old, and printed books less than 700. With the invention of calculus by Newton and Leibniz in the seventeenth century, the development of mathematics and physics has accelerated to this day. *Kant put his finger on the source of this stunning revolution: the use of rules to harness the powers of human intuition.*

Precision in science requires precise standards and conventions, in short, precise rules in both empirical and theoretical domains. The coevolution of physics and mathematics has been driven by invention and application of new rules to shape human intuition and model the physical world. The tools of technology from simple hand tools to complex machines were obviously invented. Likewise the tools of mathematics were invented, not discovered; though it may be said that theorems derived from structures built with those tools are discovered.

The vicissitudes of mathematical invention are evident in the motley assortment of mathematical tools used by physicists today, from vectors and matrices to tensors, spinors and differential forms. Far from exhibiting the unity and richness of mathematics, these “tool kits” contribute redundancy, inefficiency and obscurity [21]. A more coherent and powerful system of mathematical tools explicitly designed to integrate algebra and geometry is already well developed with a huge range of applications. Few physicists and mathematicians know about it, so an introduction to the literature is appropriate here, especially as it supports the present thesis of *mathematics by design*!

Kant himself contributed to the rule-based developments in mathematics. He was the first to formulate the abstract commutative and associative rules for addition (published by his mathematician friend Johann Schultz). Within the next century, Hermann Grassmann and W. K. Clifford provided foundations for integrating geometry, algebra and calculus into a *universal geometric calculus* that is developing with renewed vigor today. A history of *geometric algebra and calculus* is given in [20]. Its implications for the design of mathematical tools to simplify and unify the physics and mathematics curriculum are discussed in [21]. Extension to modeling spacetime, quantum mechanics and gauge theory gravity is given in [22,23].

To the question: “*What is man*?”

Aristotle answered: “*Man is a rational animal*.”
Anthropologists observe: “*Man is a tool-making animal*.”
Modeling Theory suggests: “*Man is a modeling animal*!” **Homo modelens!**

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