2. Generalized Homogeneous Coordinates for Computational Geometry†

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2.1 Introduction

The standard algebraic model for Euclidean space $\mathcal{E}^n$ is an n-dimensional real vector space $\mathcal{R}^n$ or, equivalently, a set of real coordinates. One trouble with this model is that, algebraically, the origin is a distinguished element, whereas all the points of $\mathcal{E}^n$ are identical. This deficiency in the *vector space model* was corrected early in the 19th century by removing the origin from the plane and placing it one dimension higher. Formally, that was done by introducing *homogeneous coordinates* [H91]. The vector space model also lacks adequate representation for Euclidean points or lines at infinity. We solve both problems here with a new model for $\mathcal{E}^n$ employing the tools of geometric algebra. We call it the *homogeneous model* of $\mathcal{E}^n$.

Our "new model" has its origins in the work of F. A. Wachter (1792–1817), a student of Gauss. He showed that a certain type of surface in hyperbolic geometry known as a *horosphere* is metrically equivalent to Euclidean space, so it constitutes a non-Euclidean model of Euclidean geometry. Without knowledge of this model, the technique of *comformal and projective splits* needed to incorporate it into geometric algebra were developed by Hestenes in [H91]. The *conformal split* was developed to linearize the conformal group and simplify the connection to its spin representation. The *projective split* was developed to incorporate all the advantages of homogeneous coordinates in a "coordinate-free" representation of geometrical points by vectors.

Andraes Dress and Timothy Havel [DH93] recognized the relation of the conformal split to Wachter's model as well as to classical work on *distance geometry* by Menger [M31], Blumenthal [B53, 61] and Seidel [S52, 55]. They also stressed connections to classical *invaraint theory*, for which the basics have been incorporated into geometric algebra in [HZ91] and [HS84]. The present work synthesizes all these developments and integrates conformal and projective splits into a powerful algebraic formalism for representing and manipulating

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geometric concepts. We demonstrate this power in an explicit construction of the new *homogeneous model* of $\mathcal{E}^n$, the characterization of geometric objects therein, and in the proofs of geometric theorems.

The truly new thing about our model is the algebraic formalism in which it is embedded. *This integrates the representational simplicity of synthetic geometry with the computational capabilities of analytic geometry.* As in synthetic geometry we designate points by letters a, b, $\ldots$, but we also give them algebraic properties. Thus, the outer product $a \wedge b$ represents the line determined by a and b. This notion was invented by Hermann Grassmann [G1844] and applied to projective geometry, but it was incorporated into geometric algebra only recently [HZ91]. To this day, however, it has not been used in Euclidean geometry, owing to a subtle defect that is corrected by our homogeneous model. We show that in our model $a \wedge b \wedge c$ represents the circle through the three points. If one of these points is a null vector e representing the point at infinity, then $a \wedge b \wedge e$ represents the straight line through a and b as a circle through infinity. This representation was not available to Grassmann, because he did not have the concept of null vector.

Our model also solves another problem that perplexed Grassmann thoughout his life. He was finally forced to conclude that it is impossible to define a geometrically meaningful inner product between points. The solution eluded him because it requires the concept of indefinite metric that accompanies the concept of null vector. Our model supplies an inner product $a \cdot b$ that directly represents the Euclidean distance between the points. This is a boon to distance geometry, because it greatly facilitates computation of distances among many points. Havel [H98] has used this in applications of geometric algebra to the theory of molecular conformations. The present work provides a framework for significantly advancing such applications.

We believe that our homogeneous model provides the first ideal framework for computational Euclidean geometry. The concepts and theorems of synthetic geometry can be translated into algebraic form without the unnecessary complexities of coordinates or matrices. Constructions and proofs can be done by direct computations, as needed for practical applications in computer vision and similar fields. The spin representation of conformal transformations greatly facilitates their composition and application. We aim to develop the basics and examples in sufficient detail to make applications in Euclidean geometry fairly straightforward. As a starting point, we presume familiarity with the notations and results of Chapter 1.

We have confined our analysis to Euclidean geometry, because it has the widest applicability. However, the algebraic and conceptual framework applies to geometrics of any signature. In particular, it applies to modeling spacetime geometry, but that is a matter for another time.

2.2 Minkowski Space with Conformal and Projective Splits

The real vector space $\mathcal{R}^{n,1}$ (or $\mathcal{R}^{1,n}$) is called a *Minkowski space*, after the man who introduced $\mathcal{R}^{3,1}$ as a model of spacetime. Its signature $(n,\,1)$ $(1,\,n)$ is called the *Minkowski signature*. The orthogonal group of Minkowski space is called the *Lorentz group*, the standard name in relativity theory. Its elements are called *Lorentz transformations*. The special orthogonal group of Minkowski space is called the *proper Lorentz group*, though the adjective "proper" is often dropped, especially when reflections are not of interest. A good way to remove the ambiguity is to refer to rotations in Minkowski space as *proper Lorentz rotations* composing the *proper Lorentz rotation group*.

As demonstrated in many applications to relativity physics (beginning with [H66]) the *"Minkowski algebra"* $\mathcal{R}_{n,1} = \mathcal{G}(\mathcal{R}^{n,1})$ is the ideal instrument for characterizing geometry of Minkowski space. In this paper we study its surprising utility for Euclidean geometry. For that purpose, the simplest Minkowski algebra $\mathcal{R}_{1,1}$ plays a special role.

The *Minkowski plane* $\mathcal{R}^{1,1}$ has an orthonormal basis $\{e_+, e_-\}$ defined by the properties

$$e_\pm^2 = \pm 1\,, \qquad e_+ \cdot e_- = 0\,. \tag{2.1}$$

A *null basis* $\{e_0, e\}$ can be introduced by

$$e_0 = \tfrac{1}{2}(e_- - e_+)\,, \tag{2.2a}$$

$$e = e_- + e_+\,. \tag{2.2b}$$

Alternatively, the null basis can be defined directly in terms of its properties

$$e_0^2 = e^2 = 0\,, \qquad e \cdot e_0 = -1\,. \tag{2.3}$$

A unit pseudoscalar E for $\mathcal{R}_{1,1}$ is defined by

$$E = e \wedge e_0 = e_+ \wedge e_- = e_+ e_-\,. \tag{2.4}$$

We note the properties

$$E^2 = 1\,, \qquad E^\dagger = -E\,, \tag{2.5a}$$

$$E e_\pm = e_\mp\,, \tag{2.5b}$$

$$Ee = -eE = -e\,, \qquad Ee_0 = -e_0 E = e_0\,, \tag{2.5c}$$

$$1 - E = -ee_0\,, \qquad 1 + E = -e_0 e\,. \tag{2.5d}$$

The basis vectors and null lines in $\mathcal{R}^{1,1}$ are illustrated in Fig. 2.1. It will be seen later that the asymmetry in our labels for the null vectors corresponds to an asymmetry in their geometric interpretation.

The Lorentz rotation group for the Minkowski plane is represented by the rotor

$$U_\varphi = e^{\frac{1}{2}\varphi E}\,, \tag{2.6}$$

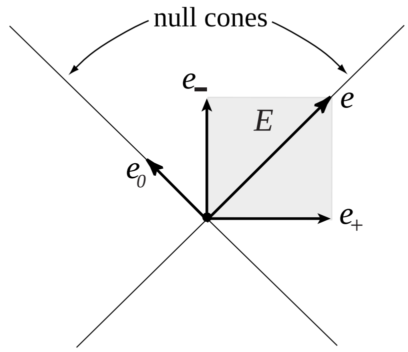

Fig 2.1. Basis vectors null lines in the Minkowski plane. The shaded area represents the unit pseudoscalar E.

where φ is a scalar parameter defined on the entire real line, and use of the symbol e to denote the exponential function will not be confused with the null vector e. Accordingly, the Lorentz rotation $\underline{U}$ of the basis vectors is given by

$$\begin{aligned} \underline{U}_\varphi e_\pm &= U_\varphi e_\pm U_\varphi^{-1} = U_\varphi^2 e_\pm \\ &= e_\pm \cosh\varphi + e_\mp \sinh\varphi \equiv e'_\pm\,, \qquad (2.7) \\ \underline{U}_\varphi e &= e^{\varphi E} e = e\, e^{-\varphi E} \equiv e'\,, \qquad (2.8) \\ \underline{U}_\varphi e_0 &= e^{\varphi E} e_0 \equiv e'_0\,. \qquad (2.9) \end{aligned}$$

The rotation is illustrated in Fig 2.2. Note that the null lines are invariant, but the null vectors are rescaled.

The complete spin group in $\mathcal{R}_{1,1}$ is

$$\mathrm{Spin}(1,1) = \{e^{\lambda E},\, E\}\,. \qquad (2.10)$$

Note that E cannot be put in exponential form, so it is not continuously connected to the identity within the group. On any vector $a \in \mathcal{R}^{1,1}$ it generates the orthogonal transformation

$$\underline{E}(a) = EaE = -a = a^*\,. \qquad (2.11)$$

Hence $\underline{E}$ is a discrete operator interchanging opposite branches of the null cone.

It is of interest to know that the Minkowski algebra $\mathcal{R}_{1,1}$ is isomorphic to the algebra $L_2(\mathcal{R})$ of real 2×2 matrices. The general linear and special linear groups have the following isomorphisms to multiplicative subgroups in $\mathcal{R}_{1,1}$

$$\{G \in \mathcal{R}_{1,1} \mid G^* G^\dagger \neq 0\} \simeq GL_2(\mathcal{R})\,, \qquad (2.12)$$

$$\{G \in \mathcal{R}_{1,1} \mid G^* G^\dagger = 1\} \simeq SL_2(\mathcal{R})\,. \qquad (2.13)$$

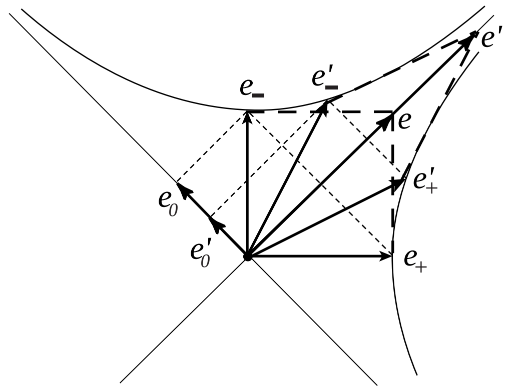

Fig 2.2. Lorentz rotations slide unit vectors along hyperbolas in the Minkowski plane, and they rescale null vectors.

The matrix representations are worked out in [H91], but they have little practical value when geometric algebra is available. The group (2.13) is a 3-parameter group whose structure is revealed by the following canonical decomposition:

$$G = K_\alpha T_\beta U_\varphi \,, \tag{2.14}$$

where $U_\varphi = U_\varphi^*$ is defined by (2.11), and

$$K_\alpha \equiv 1 + \alpha e_0 = K_\alpha^\dagger \,, \tag{2.15a}$$

$$T_\beta \equiv 1 + \beta e = T_\beta^\dagger \,. \tag{2.15b}$$

The form (2.14) holds for all values of the scalar parameters α, β, φ in the interval $[-\infty, \infty]$. Our interest in (2.14) stems from its relation to the conformal group described later.

Throughout the rest of this paper we will be working with $\mathcal{R}^{n+1,1}$, often decomposed into the direct sum

$$\mathcal{R}^{n+1,1} = \mathcal{R}^n \oplus \mathcal{R}^{1,1} \,. \tag{2.16}$$

This decomposition was dubbed a *conformal split* in [H91], because it relates to the conformal group on $\mathcal{R}^n$ in an essential way. It will be convenient to represent vectors or vector components in $\mathcal{R}^n$ by boldface letters and employ the null basis $\{e_0, e\}$ for $\mathcal{R}^{1,1}$. Accordingly, any vector $a \in \mathcal{R}^{n+1,1}$ admits the split

$$a = \mathbf{a} + \alpha e_0 + \beta e \,. \tag{2.17}$$

The conformal split is uniquely determined by the pseudoscalar E for $\mathcal{R}^{1,1}$. Let I denote the pseudoscalar for $\mathcal{R}^{n+1,1}$, then

$$\widetilde{E} = EI^{-1} = -EI^\dagger \tag{2.18}$$

is a unit pseudoscalar for $\mathcal{R}^n$, and we can express the split as

$$a = P_E(a) + P_E^{\perp}(a)\,, \tag{2.19}$$

where the projection operators P_E and $P_E^{\perp}$ are given by

$$P_E(a) = (a \cdot E)E = \alpha e_0 + \beta e \, \in \mathcal{R}^{1,1}\,, \tag{2.20a}$$

$$P_E^{\perp}(a) = (a \cdot \widetilde{E})\widetilde{E}^{\dagger} = (a \wedge E)E = \mathbf{a} \in \mathcal{R}^n\,. \tag{2.20b}$$

The Minkowski plane for $\mathcal{R}^{1,1}$ is referred to as the *E-plane*, since, as (2.20b) shows, it is uniquely determined by E. The projection $P_E^{\perp}$ can be regarded as a *rejection* from the E-plane.

It is worth noting that the conformal split was defined somewhat differently in [H91]. There the points $\mathbf{a}$ in $\mathcal{R}^n$ were identified with trivectors $(a \wedge E)E$ in (2.20b). Each of these two alternatives has its own advantages, but their representations of $\mathcal{R}^n$ are isomorphic, so the choice between them is a minor matter of convention.

The idea underlying *homogeneous coordinates* for "points" in $\mathcal{R}^n$ is to remove the troublesome origin by embedding $\mathcal{R}^n$ in a space of higher dimension. An efficient technique for doing this with geometric algebra is the *projective split* introduced in [H91]. We use it here as well. Let e be a vector in the E-plane. Then for any vector $a \in \mathcal{R}^{n+1,1}$ with $a \cdot e \neq 0$, the projective split with respect to e is defined by

$$ae = a \cdot e + a \wedge e = a \cdot e\Big(1 + \frac{a \wedge e}{a \cdot e}\Big)\,. \tag{2.21}$$

This represents vector a with the bivector $a \wedge e / a \cdot e$. The representation is independent of scale, so it is convenient to fix the scale by the condition $a \cdot e = e_0 \cdot e = -1$. This condition does not affect the components of a in $\mathcal{R}^n$. Accordingly, we refer to $e \wedge a = -a \wedge e$ as a *projective representation* for a. The classical approach to homogeneous coordinates corresponds to a projective split with respect to a non-null vector. We shall see that there are great advantages to a split with respect to a null vector. The result is a kind of "generalized" homogeneous coordinates.

A *hyperplane* $\mathcal{P}^{n+1}(n, a)$ with normal n and containing point a is the solution set of the equation

$$n \cdot (x - a) = 0\,, \qquad x \in \mathcal{R}^{n+1,1}\,. \tag{2.22}$$

As explained in Chapter 1, this can be alternatively described by

$$\widetilde{n} \wedge (x - a) = 0\,, \qquad x \in \mathcal{R}^{n+1,1}\,. \tag{2.23}$$

where $\widetilde{n} = nI^{-1}$ is the $(n+1)$-vector dual to n.

The "normalization condition" $x \cdot e = e \cdot e_0 = -1$ for a projective split with respect to the null vector e is equivalent to the equation $e \cdot (x - e_0) = 0$; thus x lie on the hyperplane

$$\mathcal{P}^{n+1}(e, e_0) = \{x \in \mathcal{R}^{n+1,1} \mid e \cdot (x - e_0) = 0\}\,. \tag{2.24}$$

This fulfills the primary objective of homogeneous coordinates by displacing the origin of $\mathcal{R}^n$ by e_0. One more condition is needed to fix x as representation for a unique $\mathbf{x}$ in $\mathcal{R}^n$.

2.3 Homogeneous Model of Euclidean Space

The set $\mathcal{N}^{n+1}$ of all null vectors in $\mathcal{R}^{n+1,1}$ is called the *null cone*. We complete our definition of *generalized homogeneous coordinates* for points in $\mathcal{R}^n$ by requiring them to be null vectors, and lie in the intersection of $\mathcal{N}^{n+1}$ with the hyperplane $\mathcal{P}^{n+1}(e, e_0)$ defined by (2.24). The resulting surface

$$\mathcal{N}_e^n = \mathcal{N}^{n+1} \cap \mathcal{P}^{n+1}(e, e_0) = \{x \in \mathcal{R}^{n+1,1} \mid x^2 = 0, \quad x \cdot e = -1\} \tag{2.25}$$

is a parabola in $\mathcal{R}^{2,1}$, and its generalization to higher dimensions is called a *horosphere* in the literature on hyperbolic geometry. Applying the conditions $x^2 = 0$ and $x \cdot e = -1$ to determine the parameters in (2.17), we get

$$x = \mathbf{x} + \tfrac{1}{2}\mathbf{x}^2 e + e_0 \,. \tag{2.26}$$

This defines a bijective mapping of $\mathbf{x} \in \mathcal{R}^n$ to $x \in \mathcal{N}_e^n$. Its inverse is the rejection (2.20b). Its projection onto the E-plane (2.20a) is shown in Fig. 2.3.

Since $\mathcal{R}^n$ is isomorphic to $\mathcal{E}^n$, so is $\mathcal{N}_e^n$, and we have proved

Theorem 1

$$\mathcal{E}^n \simeq \mathcal{N}_e^n \simeq \mathcal{R}^n \,. \tag{2.27}$$

We call $\mathcal{N}_e^n$ the *homogeneous model* of $\mathcal{E}^n$ (or $\mathcal{R}^n$), since its elements are (generalized) *homogeneous coordinates* for points in $\mathcal{E}^n$ (or $\mathcal{R}^n$). In view of their isomorphism, it will be convenient to identify $\mathcal{N}_e^n$ with $\mathcal{E}^n$ and refer to the elements of $\mathcal{N}_e^n$ simply as (*homogeneous*) *points*. The adjective homogeneous will be employed when it is necessary to distinguish these points from points in $\mathcal{R}^n$, which we refer to as *inhomogeneous points*. Our notations x and $\mathbf{x}$ in (2.26) are intended to maintain this distinction.

We have framed our discussion in terms of "homogeneous coordinates" because that is a standard concept. However, geometric algebra enables us to characterize a point as a single vector without ever decomposing a vector into a set of coordinates for representational or computational purposes. It is preferable, therefore, to speak of "homogeneous points" rather than "homogeneous coordinates."

By setting $\mathbf{x} = 0$ in (2.26) we see that e_0 is the homogeneous point corresponding to the origin of $\mathcal{R}^n$. From

$$\frac{x}{-x \cdot e_0} = e + 2\Big(\frac{\mathbf{x} + e_0}{\mathbf{x}^2}\Big) \xrightarrow[\mathbf{x}^2 \to \infty]{} e \,, \tag{2.28}$$

we see that e represents the point at infinity.

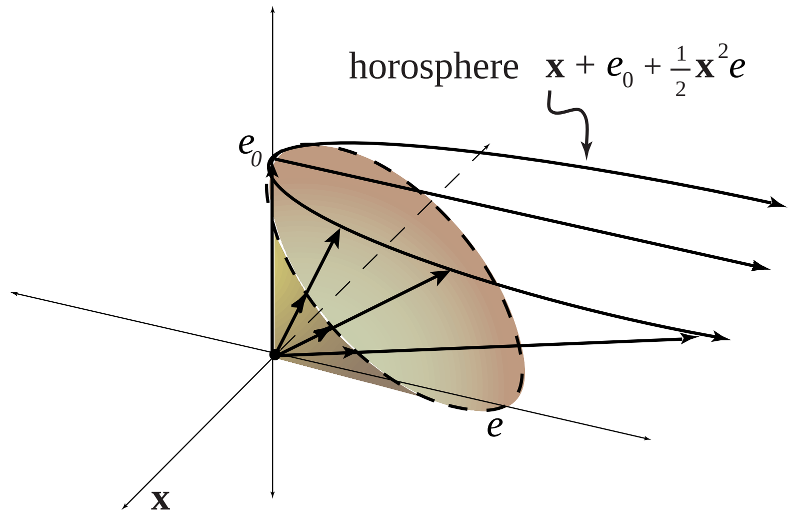

Fig 2.3. The horosphere $\mathcal{N}_e^n$ and its projection onto the E-plane.

As introduced in (2.21), the projective representation for the point (2.26) is

$$e \wedge x = \frac{e \wedge x}{-e \cdot x} = e\mathbf{x} + e \wedge e_0 \,. \tag{2.29}$$

Note that $e \wedge \mathbf{x} = e\mathbf{x} = -\mathbf{x}e$ since $e \cdot \mathbf{x} = 0$. By virtue of (2.5a) and (2.5c),

$$(e \wedge x)\, E = 1 + e\mathbf{x} \,. \tag{2.30}$$

This is identical to the representation for a point in the *affine model* of $\mathcal{E}^n$ introduced in Chapter 1. Indeed, the homogeneous model maintains and generalizes all the good features of the affine model.

Lines, planes and simplexes

Before launching into a general treatment of geometric objects, we consider how the homogeneous model characterizes the simplest objects and relations in Euclidean geometry. Using (2.26) we expand the geometric product of two points a and b as

$$ab = \mathbf{a}\mathbf{b} + (\mathbf{a} - \mathbf{b})\, e_0 - \tfrac{1}{2}\left[(\mathbf{a}^2 + \mathbf{b}^2) + (\mathbf{b}\mathbf{a}^2 - \mathbf{a}\mathbf{b}^2)\, e + (\mathbf{b}^2 - \mathbf{a}^2)E\right] . \tag{2.31}$$

From the bivector part we get

$$e \wedge a \wedge b = e \wedge (\mathbf{a} + e_0) \wedge (\mathbf{b} + e_0) = e\mathbf{a} \wedge \mathbf{b} + (\mathbf{b} - \mathbf{a})E \,. \tag{2.32}$$

From Chapter 1, we recognize $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge (\mathbf{b} - \mathbf{a})$ as the moment for a line through point $\mathbf{a}$ with tangent $\mathbf{a} - \mathbf{b}$, so $e \wedge a \wedge b$ characterizes the line completely.

Accordingly, we interpret $e \wedge a \wedge b$ as a *line passing through points* a and b, or, more specifically, as a *1-simplex with endpoints* a and b.

The scalar part of (2.31) gives us

$$a \cdot b = -\tfrac{1}{2}(\mathbf{a} - \mathbf{b})^2 \,. \tag{2.33}$$

Thus, the inner product of two homogeneous points gives directly the squared Euclidean distance between them. Since $a^2 = b^2 = 0$, we have

$$(a - b)^2 = -2\, a \cdot b = (\mathbf{a} - \mathbf{b})^2 \,. \tag{2.34}$$

Incidentally, this shows that the embedding (2.26) of $\mathcal{R}^n$ in $\mathcal{N}_e^n$ is isometric. The *squared content* of the line segment (2.32) is given by

$$\begin{aligned}(e \wedge a \wedge b)^2 &= -(b \wedge a \wedge e) \cdot (e \wedge a \wedge b) \\ &= -[\,(b \wedge a) \cdot e\,] \cdot [\,e \cdot (a \wedge b)\,] \\ &= -[a - b] \cdot [a - b] = -(a - b)^2 \,,\end{aligned} \tag{2.35}$$

which equals the negative of the squared Euclidean length of the segment, as it should. In evaluating (2.35) we used identities from Chapter 1 as well as the special properties $e^2 = 0$ and $e \cdot a = e \cdot b = -1$. Alternatively, one could use (2.32) to evaluate $(e \wedge a \wedge b)^2$ in terms of inhomogeneous points.

Again using (2.26) we find from (2.32)

$$e \wedge a \wedge b \wedge c = e\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} + E(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \,. \tag{2.36}$$

We recognize $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ as the moment of a plane with tangent $(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})$. Therefore $e \wedge a \wedge b \wedge c$ represents a plane through points a, b, c, or, more specifically, the triangle (2-simplex) with these points as vertices. The squared content of the triangle is obtained directly from

$$(e \wedge a \wedge b \wedge c)^2 = [\,(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})\,]^2 \,, \tag{2.37}$$

the negative square of twice the area of the triangle, as anticipated.

Spheres

The equation for a sphere of radius ρ centered at point $\mathbf{p}$ in $\mathcal{R}^n$ can be written

$$(\mathbf{x} - \mathbf{p})^2 = \rho^2 \,. \tag{2.38}$$

Using (2.33), we can express this as an equivalent equation in terms of homogeneous points:

$$x \cdot p = -\tfrac{1}{2}\rho^2 \,. \tag{2.39}$$

Using $x \cdot e = -1$, we can simplify this equation to

$$x \cdot s = 0 \,, \tag{2.40}$$

where

$$s = p - \tfrac{1}{2}\rho^2 e = \mathbf{p} + e_0 + \frac{\mathbf{p}^2 - \rho^2}{2}\, e\,. \tag{2.41}$$

The vector s has the properties

$$s^2 = \rho^2 > 0\,, \tag{2.42a}$$

$$e \cdot s = -1\,. \tag{2.42b}$$

From these properties the form (2.41) and center p can be recovered. Therefore, every sphere in $\mathcal{R}^n$ is completely characterized by a unique vector s in $\mathcal{R}^{n+1,1}$. According to (2.42b), s lies in the hyperplane $\mathcal{P}^{n+1,1}(e, e_0)$, but (2.42a) says that s has positive signature, so it lies outside the null cone. Our analysis shows that every such vector determines a sphere.

Alternatively, a sphere can be described by the $(n+1)$-vector $\widetilde{s} = sI^{-1}$ dual to s. Since

$$I^\dagger = (-1)^\epsilon I = -I^{-1}\,, \tag{2.43}$$

where $\epsilon = \frac{1}{2}(n+2)(n+1)$, we can express the constraints (2.42a) and (2.42b) in the form

$$s^2 = -\widetilde{s}^\dagger \widetilde{s} = \rho^2\,, \tag{2.44a}$$

$$s \cdot e = -e \cdot (\widetilde{s} I) = -(e \wedge \widetilde{s}) I = -1\,. \tag{2.44b}$$

The equation (2.40) for the sphere has the dual form

$$x \wedge \widetilde{s} = 0\,. \tag{2.45}$$

As seen later, the advantage of $\widetilde{s}$ is that it can be calculated directly from points on the sphere. Then s can be obtained by duality to find the center of the sphere. This duality of reprentations for a sphere is very powerful both computationally and conceptually. We do not know if it has been recognized before. In any case, we doubt that it has ever been expressed so simply.

Euclidean Plane Geometry

The advantages of the homogeneous model for $\mathcal{E}_2$ are best seen in an example:

Simson's Theorem. *Let ABC be a triangle and D be a point in the plane. Draw lines from D perpendicular to the three sides of the triangle and intersecting at points A_1, B_1, C_1. The points A_1, B_1, C_1 lie on a straight line if and only if D lies on the circle circumscribing triangle ABC.*

Analysis and proof of the theorem is facilitated by constructing *Simson's triangle* A_1, B_1, C_1 as shown in Fig. 4. Then the collinearity of points is linked to vanishing area of Simson's triangle.

Suspending for the moment our convention of representing vectors by lower case letters, we interpret the labels in Fig. 2.4 as homogeneous points in $\mathcal{E}^2$.

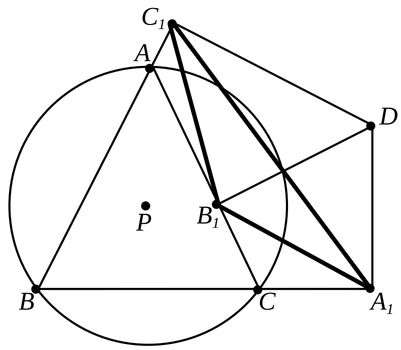

Fig 2.4. Construction of Simson's Triangle.

We have geometric algebra to express relations and facilitate analysis. We can speak of *triangle* $e \wedge A \wedge B \wedge C$ and its *side* $e \wedge A \wedge B$. This fuses the expressive advantages of synthetic geometry with the computational power of geometric algebra, as we now show.

Before proving Simson's theorem, we establish some basic results of general utility in Euclidean geometry. First, the relation between a triangle $e \wedge A \wedge B \wedge C$ and its circumcircle is

$$\widetilde{s} = A \wedge B \wedge C\,. \tag{2.46}$$

A general proof that this does indeed represent a circle (=sphere in $\mathcal{E}^2$) through the three points is given in the next section, so we take it for granted here. However, (2.46) is an unnormalized representation, so to calculate the circle radius ρ we modify (2.44a) and (2.44b) to

$$\rho^2 = \frac{s^2}{(s \cdot e)^2} = \frac{\widetilde{s}^\dagger \widetilde{s}}{(e \wedge \widetilde{s})^2} = \frac{(C \wedge B \wedge A) \cdot (A \wedge B \wedge C)}{(e \wedge A \wedge B \wedge C) \cdot (e \wedge A \wedge B \wedge C)}\,. \tag{2.47}$$

The right side of (2.47) is the ratio of two determinants, which, when expanded, express ρ^2 in terms of the distances between points, in other words, the lengths of the sides of the triangle. Recalling (2.34), the numerator gives

$$\begin{aligned}(A \wedge B \wedge C)^2 = -\begin{vmatrix} 0 & A \cdot B & A \cdot C \\ B \cdot A & 0 & B \cdot C \\ C \cdot A & C \cdot B & 0 \end{vmatrix} &= -2A \cdot B\; B \cdot C\; C \cdot A \\ &= -\tfrac{1}{4}(A - B)^2 (B - C)^2 (C - A)^2 \\ &= -\tfrac{1}{4}(\mathbf{A} - \mathbf{B})^2 (\mathbf{B} - \mathbf{C})^2 (\mathbf{C} - \mathbf{A})^2\,. \end{aligned} \tag{2.48}$$

The denominator is obtained from (2.37), which relates it to the area of the triangle and expands to

$$\begin{aligned}(e\wedge A\wedge B\wedge C)^2 &= -4(\text{area})^2\\ &= [(\mathbf{B}-\mathbf{A})\cdot(\mathbf{C}-\mathbf{A})]^2-(\mathbf{B}-\mathbf{A})^2(\mathbf{C}-\mathbf{A})^2\\ &= [(B-A)\cdot(C-A)]^2-4(A\cdot B)^2(A\cdot C)^2\,. \end{aligned}\tag{2.49}$$

By normalizing $A\wedge B\wedge C$ and taking its dual, we find the center P of the circle from (2.41); thus

$$\frac{-(A\wedge B\wedge C)^{\sim}}{(e\wedge A\wedge B\wedge C)^{\sim}} = P-\tfrac{1}{2}\rho^2\, e\,. \tag{2.50}$$

This completes our characterization of the intrinsic properties of a triangle.

To relate circle $A\wedge B\wedge C$ to a point D, we use

$$(A\wedge B\wedge C)\vee D = (A\wedge B\wedge C)^{\sim}\cdot D = -(A\wedge B\wedge C\wedge D)^{\sim}$$

with (2.50) to get

$$A\wedge B\wedge C\wedge D = \frac{\rho^2-\delta^2}{2}\, e\wedge A\wedge B\wedge C\,, \tag{2.51}$$

where

$$\delta^2 = -2P\cdot D \tag{2.52}$$

is the squared distance between D and P. According to (2.45), the left side of (2.51) vanishes when D is on the circle, in conformity with $\delta^2=\rho^2$ on the right side of (2.51).

To construct the Simson triangle algebraically, we need to solve the problem of finding the "perpendicular intersection" B_1 of point D on line $e\wedge A\wedge C$ (Fig. 2.4). Using inhomogeneous points we can write the condition for perpendicularity as

$$(\mathbf{B}_1-\mathbf{D})\cdot(\mathbf{C}-\mathbf{A}) = 0\,. \tag{2.53}$$

Therefore

$$(\mathbf{B}_1-\mathbf{D})(\mathbf{C}-\mathbf{A}) = (\mathbf{B}_1-\mathbf{D})\wedge(\mathbf{C}-\mathbf{A}) = (\mathbf{A}-\mathbf{D})\wedge(\mathbf{C}-\mathbf{A})\,.$$

Dividing by $(\mathbf{C}-\mathbf{A})$,

$$\begin{aligned}\mathbf{B}_1-\mathbf{D} &= [(\mathbf{A}-\mathbf{D})\wedge(\mathbf{C}-\mathbf{A})]\cdot(\mathbf{C}-\mathbf{A})^{-1}\\ &= \mathbf{A}-\mathbf{D}-(\mathbf{A}-\mathbf{D})\cdot(\mathbf{C}-\mathbf{A})^{-1}(\mathbf{C}-\mathbf{A})\,.\end{aligned}\tag{2.54}$$

Therefore

$$\mathbf{B}_1 = \mathbf{A}+\frac{(\mathbf{D}-\mathbf{A})\cdot(\mathbf{C}-\mathbf{A})}{(\mathbf{C}-\mathbf{A})^2}\,(\mathbf{C}-\mathbf{A})\,. \tag{2.55}$$

We can easily convert this to a relation among homogeneous points. However, we are only interested here in Simson's triangle $e \wedge A_1 \wedge B_1 \wedge C_1$, which by (2.36) can be represented in the form

$$\begin{aligned} e \wedge A_1 \wedge B_1 \wedge C_1 &= E(\mathbf{B}_1 - \mathbf{A}_1) \wedge (\mathbf{C}_1 - \mathbf{A}_1) \\ &= E(\mathbf{A}_1 \wedge \mathbf{B}_1 + \mathbf{B}_1 \wedge \mathbf{C}_1 + \mathbf{C}_1 \wedge \mathbf{A}_1)\,. \end{aligned} \tag{2.56}$$

Calculations are simplified considerably by identifying $\mathbf{D}$ with the origin in $\mathcal{R}^n$, which we can do without loss of generality. Then equation (2.52) becomes $\delta^2 = -2P \cdot D = \mathbf{p}^2$. Setting $\mathbf{D} = 0$ in (2.55) and determining the analogous expressions for $\mathbf{A}_1$ and $\mathbf{C}_1$, we insert the three points into (2.56) and find, after some calculation,

$$e \wedge A_1 \wedge B_1 \wedge C_1 = \Big(\frac{\rho^2 - \delta^2}{4\rho^2}\Big)\, e \wedge A \wedge B \wedge C\,. \tag{2.57}$$

The only tricky part of the calculation is getting the coefficient on the right side of (2.57) in the form shown. To do that the expanded form for ρ^2 in (2.47) to (2.49) can be used.

Finally, combining (2.57) with (2.51) we obtain the identity

$$e \wedge A_1 \wedge B_1 \wedge C_1 = \frac{A \wedge B \wedge C \wedge D}{2\rho^2}\,. \tag{2.58}$$

This proves Simson's theorem, for the right side vanishes if and only if D is on the circle, while the left side vanishes if and only if the three points lie on the same line.

2.4 Euclidean Spheres and Hyperspheres

A hyperplane through the origin is called a *hyperspace*. A hyperspace $\mathcal{P}^{n+1}(s)$ in $\mathcal{R}^{n+1,1}(s)$ with Minkowski signature is called a *Minkowski hyperspace*. Its normal s must have positive.

Theorem 2 *The intersection of any Minkowski hyperspace $\mathcal{P}^{n+1}(s)$ with the horosphere $\mathcal{N}_e^{n+1}(s) \simeq \mathcal{E}^n$ is a sphere or hyperplane*

$$\mathcal{S}(s) = \mathcal{P}^{n+1}(s) \cap \mathcal{N}_e^{n+1} \tag{2.59}$$

in $\mathcal{E}^n$ (or $\mathcal{R}^n$), and every Euclidean sphere or hyperplane can be obtained in this way. $\mathcal{S}(s)$ is a sphere if $e \cdot s < 0$ or a hyperplane if $e \cdot s = 0$.

Corollary. *Every Euclidean sphere or hyperplane can be represented by a vector s (unique up to scale) with $s^2 > 0$ and $s \cdot e \leq 0$.*

From our previous discussion we know that the sphere $\mathcal{S}(s)$ has radius ρ given by

$$\rho^2 = \frac{s^2}{(s \cdot e)^2}\,, \tag{2.60}$$

and it is centered at point

$$p = \frac{s}{-s \cdot e} + \tfrac{1}{2}\rho^2 e\,. \tag{2.61}$$

Therefore, with the normalization $s \cdot e = -1$, each sphere is represented by a unique vector. With this normalization, the set $\{\mathbf{x} = P_E^{\perp}(x) \in \mathcal{R}^n | x \cdot s > 0\}$ represents the interior of the sphere, and we refer to (2.61) as the *standard form* for the representation of a sphere by vector s.

To prove Theorem 2, it suffices to analyze the two special cases. These cases are distinguished by the identity

$$(s \cdot e)^2 = (s \wedge e)^2 \geq 0\,, \tag{2.62}$$

which follows from $e^2 = 0$. We have already established that $(e \cdot s)^2 > 0$ characterizes a sphere. For the case $e \cdot s = 0$, we observe that the component of s in $\mathcal{R}^n$ is given by

$$\mathbf{s} = P_E^{\perp}(s) = (s \wedge E)E = s + (s \cdot e_0)\,e\,. \tag{2.63}$$

Therefore

$$s = |\,s\,|(\mathbf{n} + e\,\delta)\,, \tag{2.64}$$

where $\mathbf{n}^2 = 1$ and $\delta = s \cdot e_0/|\,s\,|$. Set $|\,s\,| = 1$. The equation for a point x on the surface $\mathcal{S}(s)$ is then

$$x \cdot s = \mathbf{n} \cdot \mathbf{x} - \delta = 0\,. \tag{2.65}$$

This is the equation for a hyperplane in $\mathcal{R}^n$ with unit normal $\mathbf{n}$ and signed distance δ from the origin. Since $x \cdot e = 0$, the "point at infinity" e lies on $\mathcal{S}(s)$. Therefore, a *hyperplane $\mathcal{E}^n$ can be regarded as a sphere that "passes through" the point at infinity.*

With $|\,s\,| = 1$, we refer to (2.64) as the *standard form* for representation of a hyperplane by vector s.

Theorem 3 *Given homogeneous points a_0, a_1, $a_2, \ldots,$ a_n "in" $\mathcal{E}^n$ such that*

$$\widetilde{s} = a_0 \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n \neq 0\,, \tag{2.66}$$

then the $(n+1)$-blade $\widetilde{s}$ represents a Euclidean sphere if

$$(e \wedge \widetilde{s})^2 \neq 0\,. \tag{2.67}$$

or a hyperplane if

$$(e \wedge \widetilde{s})^2 = 0\,. \tag{2.68}$$

A point x is on the sphere/hyperplane $\mathcal{S}(s)$ if and only if

$$x \wedge \widetilde{s} = 0\,. \tag{2.69}$$

Since (2.66) is a condition for linear independence, we have the converse theorem that every $\mathcal{S}(s)$ is uniquely determined by $n+1$ linearly independent points.

By duality, Theorem 3 is an obvious consequence of Theorem 2 where $\widetilde{s}$ is dual to the normal s of the hyperspace $\mathcal{P}^{n+1}(s)$, so it is a tangent for the hyperspace.

For a hyperplane, we can always employ the point at infinity so the condition (2.66) becomes

$$\widetilde{s} = e \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n \neq 0 \,. \tag{2.70}$$

Therefore only n linearly independent finite points are needed to define a hyperplane in $\mathcal{E}^n$.

2.5 r-dimensional Spheres, Planes and Simplexes

We have seen that $(n+1)$-blades of Minkowski signature in $\mathcal{R}^{n+1,1}$ represent spheres and hyperplanes in $\mathcal{R}^n$, so the following generalization is fairly obvious

Theorem 4 *For $2 \leq r \leq n+1$, every r-blade A_r of Minkowski signature in $\mathcal{R}^{n+1,1}$ represents an $(r-2)$-dimensional sphere in $\mathcal{R}^n$ (or $\mathcal{E}^n$).*

There are three cases to consider:

Case 1. $e \wedge A_r = e_0 \wedge A_r = 0$, A_r represents an $(r-2)$-plane through the origin in $\mathcal{R}^n$ with *standard form*

$$A_r = E\mathbf{I}_{r-2} \,, \tag{2.71}$$

where $\mathbf{I}_{r-2}$ is unit tangent for the plane.

Case 2. A_r represents an $(r-2)$-plane when $e \wedge A_r = 0$ and

$$A_{r+1} = e_0 \wedge A_r \neq 0 \,. \tag{2.72}$$

We can express A_r as the dual of a vector s with respect to A_{r+1}:

$$A_r = sA_{r+1} = (-1)^{\epsilon}\, \widetilde{s} \vee A_{r+1} \,. \tag{2.73}$$

In this case $e \cdot s = 0$ but $s \cdot e_0 \neq 0$, so we can write s in the standard form $s = \mathbf{n} + \delta e$ for the hyperplane $\widetilde{s}$ with unit normal $\mathbf{n}$ in $\mathcal{R}^n$ and $\mathbf{n}$-distance δ from the origin. Normalizing A_{r+1} to unity, we can put A_r into the *standard form*

$$A_r = (\mathbf{n} + e\delta)E\mathbf{I}_{r-1} = E\mathbf{n}\mathbf{I}_{r-1} + e\delta\mathbf{I}_{r-1} \,. \tag{2.74}$$

This represents an $(r-2)$-plane with unit tangent $\mathbf{n}\mathbf{I}_{r-1} = \mathbf{n}\cdot\mathbf{I}_{r-1}$ and moment $\delta\mathbf{I}_{r-1}$. Its directance from the origin is the vector $\delta\mathbf{n}$.

As a corollary to (2.74), the r-plane passing through point $\mathbf{a}$ in $\mathcal{R}^n$ with unit r-blade $\mathbf{I}_r$ as tangent has the *standard form*

$$A_{r+1} = e \wedge a \wedge \mathbf{I}_r \,, \tag{2.75}$$

where $\mathbf{a} = P_E^{\perp}(a)$ is the inhomogeneous point.

Case 3. A_r represents an $(r-2)$-dimensional sphere if

$$A_{r+1} \equiv e \wedge A_r \neq 0 \,. \tag{2.76}$$

The vector

$$s = A_r A_{r+1}^{-1} \tag{2.77}$$

has positive square and $s \cdot e \neq 0$, so its dual $\widetilde{s} = sI^{-1}$ represents an $(n-1)$-dimensional sphere

$$A_r = sA_{r+1} = (\widetilde{s}I) \cdot A_{r+1} = (-1)^{\epsilon}\, \widetilde{s} \vee A_{r+1} \,, \tag{2.78}$$

where the (inessential) sign is determined by (2.43). As shown below, condition (2.76) implies that A_{r+1} represents an $(r-1)$-plane in $\mathcal{R}^n$. Therefore the meet product $\widetilde{s} \vee A_{r+1}$ in (2.78) expresses the $(r-2)$-sphere A_r as the intersection of the $(n-1)$-sphere $\widetilde{s}$ with the $(r-1)$-plane A_{r+1}.

With suitable normalization, we can write $s = c - \frac{1}{2}\rho^2 e$ where c is the center and ρ is the radius of sphere $\widetilde{s}$. Since $s \wedge A_{r+1} = e \wedge A_{r+1} = 0$, the sphere A_r is also centered at point c and has radius ρ.

Using (2.74) for the standard form of A_{r+1}, we can represent an $(r-2)$-sphere on a plane in the *standard form*

$$A_r = (c - \tfrac{1}{2}\rho^2 e) \wedge (\mathbf{n} + e\,\delta)E\mathbf{I}_r \,, \tag{2.79}$$

where $|\,\mathbf{I}_r\,| = 1$, $\mathbf{c} \wedge \mathbf{I}_r = \mathbf{n} \wedge \mathbf{I}_r = 0$ and $\mathbf{c} \cdot \mathbf{n} = \delta$.

In particular, we can represent an $(r-2)$-sphere in a space in the *standard form*

$$A_r = (c - \tfrac{1}{2}\rho^2 e)E\mathbf{I}_{r-1} \,, \tag{2.80}$$

where $E = e \wedge e_0$ and $\mathbf{I}_{r-1}$ is a unit $(r-1)$-blade in $\mathcal{R}_n$. In (2.80) the factor $E\mathbf{I}_{r-1}$ has been normalized to unit magnitude. Both (2.78) and (2.80) express A_r as the dual of vector s with respect to A_{r+1}. Indeed, for $r = n+1$, $\mathbf{I}_n$ is a unit pseudoscalar for $\mathcal{R}_n$, so (2.78) and (2.80) give the dual form $\widetilde{s}$ that we found for spheres in the preceding section.

This completes our classification of standard representations for spheres and planes in $\mathcal{E}^n$.

Simplexes and spheres

Now we examine geometric objects determined by linearly independent homogeneous points $a_0, a_1, \ldots, a_r$, with $r \le n$ so that $a_0 \wedge a_1 \wedge \cdots \wedge a_r \neq 0$. Introducing inhomogeneous points by (2.26), a simple computation gives the *expanded form*

$$a_0 \wedge a_1 \wedge \cdots \wedge a_r = \mathbf{A}_r + e_0 \mathbf{A}_r^+ + \tfrac{1}{2} e \mathbf{A}_r^- - \tfrac{1}{2} E \mathbf{A}_r^\pm , \tag{2.81}$$

where, for want of a better notation,

$$\begin{aligned}
\mathbf{A}_r &= \mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_r, \\
\mathbf{A}_r^+ &= \sum_{i=0}^{r} (-1)^i \mathbf{a}_0 \wedge \cdots \wedge \check{\mathbf{a}}_i \wedge \cdots \wedge \mathbf{a}_r = (\mathbf{a}_1 - \mathbf{a}_0) \wedge \cdots \wedge (\mathbf{a}_r - \mathbf{a}_0), \\
\mathbf{A}_r^- &= \sum_{i=0}^{r} (-1)^i \mathbf{a}_i^2 \mathbf{a}_0 \wedge \cdots \wedge \check{\mathbf{a}}_i \wedge \cdots \wedge \mathbf{a}_r, \\
\mathbf{A}_r^\pm &= \sum_{i=0}^{r} \sum_{j=i+1}^{r} (-1)^{i+j} (\mathbf{a}_i^2 - \mathbf{a}_j^2) \mathbf{a}_0 \wedge \cdots \wedge \check{\mathbf{a}}_i \wedge \cdots \wedge \check{\mathbf{a}}_j \wedge \cdots \wedge \mathbf{a}_r.
\end{aligned} \tag{2.82}$$

Theorem 5 *The expanded form* (2.81)

(1) *determines an r-simplex if $\mathbf{A}_r \neq 0$,*

(2) *represents an $(r-1)$-simplex in a plane through the origin if $\mathbf{A}_r^+ = \mathbf{A}_r^- = 0$,*

(3) *represents an $(r-1)$-sphere if and only if $\mathbf{A}_r^+ \neq 0$.*

We establish and analyze each of these three cases in turn.

From our study of simplexes in Chapter 1, we recognize $\mathbf{A}_r$ as the *moment* of a simplex with *boundary* (or *tangent*) $\mathbf{A}_r^+$. Therefore,

$$e \wedge a_0 \wedge a_1 \wedge \cdots \wedge a_r = e\mathbf{A}_r + E\mathbf{A}_r^+ \tag{2.83}$$

represents an r-simplex. The *volume* (or *content*) of the simplex is $k! \,|\, \mathbf{A}_r^+ \,|$, where

$$\begin{aligned}
|\, \mathbf{A}_r^+ \,|^2 = (\mathbf{A}_r^+)^\dagger \mathbf{A}_r^+ &= -(a_r \wedge \cdots \wedge a_0 \wedge e) \cdot (e \wedge a_0 \wedge \cdots \wedge a_r) \\
&= -(-\tfrac{1}{2})^r \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & d_{ij}^2 & \\ 1 & & & \end{vmatrix}
\end{aligned} \tag{2.84}$$

and $d_{ij} = |\, \mathbf{a}_i - \mathbf{a}_j \,|$ is the pairwise interpoint distance. The determinant on the right side of (2.84) is called the *Cayley-Menger determinant*, because Cayley found it as an expression for volume in 1841, and nearly a century later Menger [M31] used it to reformulate Euclidean geometry with the notion of interpoint distance as a primitive.

Comparison of (2.83) with (2.74) gives the directed distance from the origin in $\mathcal{R}^n$ to the plane of the simplex in terms of the points:

$$\delta\mathbf{n} = \mathbf{A}_r(\mathbf{A}_r^+)^{-1}. \tag{2.85}$$

Therefore, the squared distance is given by the ratio of determinants:

$$\delta^2 = \frac{|\,\mathbf{A}_r\,|^2}{|\,\mathbf{A}_r^+\,|^2} = \frac{(\mathbf{a}_r \wedge \cdots \wedge \mathbf{a}_0)\cdot(\mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_r)}{(\bar{\mathbf{a}}_r \wedge \cdots \wedge \bar{\mathbf{a}}_1)\cdot(\bar{\mathbf{a}}_1 \wedge \cdots \wedge \bar{\mathbf{a}}_r)}, \tag{2.86}$$

where $\bar{\mathbf{a}}_i = \mathbf{a}_i - \mathbf{a}_0$ for $i = 1, \ldots, r$, and the denominator is an alternative to (2.84).

When $\mathbf{A}_r^+ = \mathbf{A}_r^- = 0$, (2.81) reduces to

$$a_0 \wedge \cdots \wedge a_r = -\tfrac{1}{2}E\mathbf{A}_r^{\pm}. \tag{2.87}$$

Comparing with (2.83) we see that this degenerate case represents an $(r-1)$-simplex with volume $\frac{1}{2}k!|\,A_r^{\pm}\,|$ in an $(r-1)$-plane through the origin. To get an arbitrary $(r-1)$-simplex from $a_0 \wedge \cdots \wedge a_r$ we must place one of the points, say a_0, at ∞. Then we have $e \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r$, which has the same form as (2.83).

We get more insight into the expanded form (2.81) by comparing it with the standard forms (2.79), (2.80) for a sphere. When $\mathbf{A}_r = 0$, then $\mathbf{A}_r^+ \neq 0$ for $a_0 \wedge \cdots \wedge a_r$ to represent a sphere. Since

$$a_0 \wedge \cdots \wedge a_r = -[e_0 - \tfrac{1}{2}e\mathbf{A}_r^-(\mathbf{A}_r^+)^{-1} + \tfrac{1}{2}\mathbf{A}_r^{\pm}(\mathbf{A}_r^+)^{-1}]E\mathbf{A}_r^+,$$

we find that the sphere is in the space represented by $E\mathbf{A}_r^+$, with center and squared radius

$$\mathbf{c} = \tfrac{1}{2}\mathbf{A}_r^{\pm}(\mathbf{A}_r^+)^{-1}, \tag{2.88a}$$

$$\rho^2 = \mathbf{c}^2 + \mathbf{A}_r^-(\mathbf{A}_r^+)^{-1}. \tag{2.88b}$$

When $\mathbf{A}_r \neq 0$, then $\mathbf{A}_r^+ \neq 0$ because of (2.92b) below. Since

$$a_0 \wedge \cdots \wedge a_r = \frac{(\mathbf{A}_r + e_0\mathbf{A}_r^+ + \frac{1}{2}e\mathbf{A}_r^- - \frac{1}{2}E\mathbf{A}_r^{\pm})(e\mathbf{A}_r + E\mathbf{A}_r^+)^\dagger}{(e\mathbf{A}_r + E\mathbf{A}_r^+)^\dagger(e\mathbf{A}_r + E\mathbf{A}_r^+)}(e\mathbf{A}_r + E\mathbf{A}_r^+),$$

and the numerator equals

$$\mathbf{A}_r^+(\mathbf{A}_r^+)^\dagger\left[e_0 + \frac{2\mathbf{A}_r^+(\mathbf{A}_r)^\dagger + \mathbf{A}_r^{\pm}(\mathbf{A}_r^+)^\dagger}{2\mathbf{A}_r^+(\mathbf{A}_r^+)^\dagger} + \frac{2\mathbf{A}_r(\mathbf{A}_r)^\dagger - \mathbf{A}_r^-(\mathbf{A}_r^+)^\dagger}{2\mathbf{A}_r^+(\mathbf{A}_r^+)^\dagger}e\right],$$

we find that the sphere is on the plane represented by $e\mathbf{A}_r + E\mathbf{A}_r^+$, with center and squared radius

$$\mathbf{c} = \frac{2(\mathbf{A}_r^+)^{-1}(\mathbf{A}_r)^\dagger + \mathbf{A}_r^{\pm}(\mathbf{A}_r^+)^\dagger}{2\mathbf{A}_r^+(\mathbf{A}_r^+)^\dagger}, \tag{2.89a}$$

$$\rho^2 = \mathbf{c}^2 + \frac{\mathbf{A}_r^-(\mathbf{A}_r^+)^\dagger - 2\mathbf{A}_r(\mathbf{A}_r)^\dagger}{\mathbf{A}_r^+(\mathbf{A}_r^+)^\dagger}. \tag{2.89b}$$

We see that (2.89a), (2.89b) congrue with (2.88a), (2.88b) when $\mathbf{A}_r = 0$.

Having shown how the expanded form (2.81) represents spheres or planes of any dimension, let us analyze relation among the $\mathbf{A}$'s. In (2.82) $\mathbf{A}_r^+$ is already represented as a blade; when $\mathbf{a}_i \neq 0$ for all i, the analogous representation for $\mathbf{A}_r^-$ is

$$\mathbf{A}_r^- = \Pi_r(\mathbf{a}_1^{-1} - \mathbf{a}_0^{-1}) \wedge (\mathbf{a}_2^{-1} - \mathbf{a}_0^{-1}) \wedge \cdots \wedge (\mathbf{a}_r^{-1} - \mathbf{a}_0^{-1}), \tag{2.90}$$

where

$$\Pi_r = \mathbf{a}_0^2 \mathbf{a}_1^2 \cdots \mathbf{a}_r^2 . \tag{2.91}$$

From this we see that $\mathbf{A}_r^+$ and $\mathbf{A}_r^-$ are interchanged by inversions $\mathbf{a}_i \to \mathbf{a}_i^{-1}$, of all inhomogeneous points.

Using the notation for the boundary of a simplex from Chapter 1, we have

$$\mathbf{A}_r^+ \;=\; \partial\!\!\!/ \mathbf{A}_r , \qquad \mathbf{A}_r^- / \Pi_r = \partial\!\!\!/ (\mathbf{A}_r / \Pi_r) , \tag{2.92a}$$

$$\mathbf{A}_r^\pm \;=\; -\partial\!\!\!/ \mathbf{A}_r^- , \qquad \mathbf{A}_r^\pm / \Pi_r = \partial\!\!\!/ (\mathbf{A}_r^+ / \Pi_r) . \tag{2.92b}$$

An immediate corollary is that all $\mathbf{A}$'s are blades, and if $\mathbf{A}_r^\pm = 0$ then all other $\mathbf{A}$'s are zero.

If $\mathbf{A}_r \neq 0$, then we have the following relation among the four $\mathbf{A}$'s:

$$\mathbf{A}_r^+ \vee \mathbf{A}_r^- = -\widetilde{\mathbf{A}}_r \mathbf{A}_r^\pm , \tag{2.93}$$

where the meet and dual are defined in $\mathcal{G}(\mathbf{A}_r)$. Hence when $\mathbf{A}_r \neq 0$, the vector spaces defined by $\mathbf{A}_r^+$ and $\mathbf{A}_r^-$ intersect and the intersection is the vector space defined by $\mathbf{A}_r^\pm$.

Squaring (2.81) we get

$$\begin{aligned} |\, a_0 \wedge \cdots \wedge a_r \,|^2 &= \det(a_i \cdot a_j) = (-\tfrac{1}{2})^{r+1} \det(|\, \mathbf{a}_i - \mathbf{a}_j \,|^2) \\ &= |\, \mathbf{A}_r \,|^2 - (\mathbf{A}_r^+)^\dagger \cdot \mathbf{A}_r^- - \tfrac{1}{4} |\, \mathbf{A}_r^\pm \,|^2 . \end{aligned} \tag{2.94}$$

For $r = n+1$, $\mathbf{A}_r$ vanishes and we obtain

Ptolemy's Theorem: *Let $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{n+1}$ be points in $\mathcal{R}^n$, then they are on a sphere or a hypersphere if and only if* $\det(|\, \mathbf{a}_i - \mathbf{a}_j \,|^2)_{(n+2)\times(n+2)} = 0$.

2.6 Relation among Spheres and Hyperplanes

In Section 4 we learned that every sphere or hyperplane in $\mathcal{E}^n$ is *uniquely* represented by some vector s with $s^2 > 0$ or by its dual $\widetilde{s}$. It will be convenient, therefore, to use s or $\widetilde{s}$ as *names* for the surface they represent. We also learned that spheres and hyperplanes are distinguished, respectively, by the conditions $s \cdot e > 0$ and $s \cdot e = 0$, and the latter tells us that a hyperplane can be regarded as a sphere through the point at infinity. This intimate relation between spheres and hyperplanes makes it easy to analyze their common properties.

A main advantage of the representation by s and $\widetilde{s}$ is that it can be used directly for algebraic characterization of both qualitative and quantitative properties of surfaces without reference to generic points on the surfaces. In this section we present important examples of qualitative relations among spheres and hyperplanes that can readily be made quantitative. The simplicity of these relations and their classifications should be of genuine value in computational geometry, especially in problems of constraint satisfaction.

Intersection of spheres and hyperplanes

Let $\widetilde{s}_1$ and $\widetilde{s}_2$ be two different spheres or hyperplanes of $\mathcal{R}^n$ (or $\mathcal{E}^n$). Both $\widetilde{s}_1$ and $\widetilde{s}_2$ are tangent $(n+1)$-dimensional Minkowski subspaces of $\mathcal{R}^{n+1,1}$. These subspaces intersect in an n-dimensional subspace with n-blade tangent given algebraically by the *meet* product $\widetilde{s}_1 \vee \widetilde{s}_2$ defined in Chapter 1. This illustrates how the homogeneous model of $\mathcal{E}^n$ reduces the computations of intersections of spheres and planes of any dimension to intersections of linear subspaces in $\mathcal{R}^{n+1,1}$, which are computed with the meet product.

To classify topological relations between two spheres or hyperplanes, it will be convenient to work with the dual of the meet:

$$(\widetilde{s}_1 \vee \widetilde{s}_2)^{\sim} = s_1 \wedge s_2 \,. \tag{2.95}$$

There are three cases corresponding to the possible signatures of $s_1 \wedge s_2$:

Theorem 6 *Two spheres or hyperplanes $\widetilde{s}_1$, $\widetilde{s}_2$ intersect, are tangent or parallel, or do not intersect if and only if $(s_1 \wedge s_2)^2 <, =, > 0$, respectively.*

Let us examine the various cases in more detail.
When $\widetilde{s}_1$ and $\widetilde{s}_2$ are both spheres, then

- *if they intersect*, the intersection $(s_1 \wedge s_2)^{\sim}$ is a sphere, as $e \wedge (s_1 \wedge s_2)^{\sim} \neq 0$. The center and radius of the intersection are the same with those of the sphere $(P_{s_1 \wedge s_2}(e))^{\sim}$. The intersection lies on the hyperplane $(e \cdot (s_1 \wedge s_2))^{\sim}$.
- *if they are tangent*, the tangent point is proportional to the null vector $P^{\perp}_{s_1}(s_2) = (s_2 \wedge s_1)s_1^{-1}$.
- *if they do not intersect*, there are two points $\mathbf{a}, \mathbf{b} \in \mathcal{R}^n$, called *Poncelet points* [S88], which are inversive to each other with respect to both spheres $\widetilde{s}_1$ and $\widetilde{s}_2$. The reason is, since $s_1 \wedge s_2$ is Minkowski, it contains two noncollinear null vectors $|s_1 \wedge s_2| s_1 \pm |s_1||s_2| P^{\perp}_{s_1}(s_2)$, which correspond to $\mathbf{a}, \mathbf{b} \in \mathcal{R}^n$ respectively. Let $s_i = \lambda_i a + \mu_i b$, where λ_i, μ_i are scalars. Then the inversion of a homogeneous point a with respect to the sphere $\mathbf{s}_i$ gives the point $\underline{s_i}\, a = (-\mu_i/\lambda_i) b$, as shown in the section on conformal transformations.

When $\widetilde{s}_1$ is a hyperplane and $\widetilde{s}_2$ is a sphere, then

- *if they intersect*, the intersection $(s_1\wedge s_2)^\sim$ is a sphere, since $e\wedge(s_1\wedge s_2)^\sim \neq 0$. The center and radius of the intersection are the same with those of the sphere $(P_{s_1}^\perp(s_2))^\sim$.
- *if they are tangent*, the tangent point corresponds to the null vector $P_{s_1}^\perp(s_2)$.

  When a sphere $\widetilde{s}$ and a point a on it is given, the tangent hyperplane of the sphere at a is $(s + s\cdot ea)^\sim$.
- *if they do not intersect*, there are two points $\mathbf{a}, \mathbf{b} \in \mathcal{R}^n$ as before, called *Poncelet points* [S88], which are symmetric with respect to the hyperplane $\widetilde{s}_1$ and also inversive to each other with respect to the sphere $\widetilde{s}_2$.

When $\widetilde{s}_1$ and $\widetilde{s}_2$ are both hyperplanes, they always intersect or are parallel, as $(s_1\wedge s_2)^\sim$ always contains e, and therefore cannot be Euclidean. For the two hyperplanes,

- *if they intersect*, the intersection $(s_1\wedge s_2)^\sim$ is an $(n-2)$-plane. When both $\widetilde{s}_1$ and $\widetilde{s}_2$ are hyperspaces, the intersection corresponds to the $(n-2)$-space $(s_1\wedge s_2)\mathbf{I}_n$ in $\mathcal{R}^n$, where $\mathbf{I}_r$ is a unit pseudoscalar of $\mathcal{R}^n$; otherwise the intersection is in the hyperspace $(e_0\cdot(s_1\wedge s_2))^\sim$ and has the same normal and distance from the origin as the hyperplane $(P_{s_1\wedge s_2}(e_0))^\sim$.
- *if they are parallel*, the distance between them is $|e_0\cdot P_{s_2}^\perp(s_1)|/|s_1|$.

Now let us examine the geometric significance of the inner product $s_1\cdot s_2$. For spheres and hyperspaces $\widetilde{s}_1, \widetilde{s}_2$, the scalar $s_1\cdot s_2/|s_1||s_2|$ is called the *inversive product* [I92] and denoted by $s_1 * s_2$. Obviously, it is invariant under orthogonal transformations in $\mathcal{R}^{n+1,1}$, and

$$(s_1 * s_2)^2 = 1 + \frac{(s_1\wedge s_2)^2}{s_1^2 s_2^2}\,. \tag{2.96}$$

Let us assume that $\widetilde{s}_1$ and $\widetilde{s}_2$ are normalized to standard form. Following [I92, p. 40, 8.7], when $\widetilde{s}_1$ and $\widetilde{s}_2$ intersect, let $\mathbf{a}$ be a point of intersection, and let $\mathbf{m}_i$, $i = 1, 2$, be the respective outward unit normal vector of $\widetilde{s}_i$ at $\mathbf{a}$ if it is a sphere, or the negative of the unit normal vector in the standard form of $\widetilde{s}_i$ if it is a hyperplane; then

$$s_1 * s_2 = \mathbf{m}_1\cdot\mathbf{m}_2. \tag{2.97}$$

The above conclusion is proved as follows: For $i = 1, 2$, when $\widetilde{s}_i$ represents a sphere with standard form $s_i = c_i - \frac{1}{2}\rho_i^2 e$ where c_i is its center, then

$$s_1 * s_2 = \frac{\rho_1^2+\rho_2^2-|\mathbf{c}_1-\mathbf{c}_2|^2}{2\rho_1\rho_2}, \tag{2.98}$$

$$\mathbf{m}_1\cdot\mathbf{m}_2 = \frac{(\mathbf{a}-\mathbf{c}_1)}{|\mathbf{a}-\mathbf{c}_1|}\cdot\frac{(\mathbf{a}-\mathbf{c}_2)}{|\mathbf{a}-\mathbf{c}_2|} = \frac{\rho_1^2+\rho_2^2-|\mathbf{c}_1-\mathbf{c}_2|^2}{2\rho_1\rho_2}\,. \tag{2.99}$$

When s_2 is replaced by the standard form $\mathbf{n}_2 + \delta_2 e$ for a hyperplane, then

$$s_1 * s_2 = \frac{\mathbf{c}_1 \cdot \mathbf{n}_2 - \delta_2}{\rho_1}, \tag{2.100}$$

$$\mathbf{m}_1 \cdot \mathbf{m}_2 = \frac{(\mathbf{a} - \mathbf{c}_1)}{|\mathbf{a} - \mathbf{c}_1|} \cdot (-\mathbf{n}_2) = \frac{\mathbf{c}_1 \cdot \mathbf{n}_2 - \delta_2}{\rho_1}; \tag{2.101}$$

For two hyperspheres $s_i = \mathbf{n}_i + \delta_i f$; then

$$s_1 * s_2 = \mathbf{n}_1 \cdot \mathbf{n}_2, \tag{2.102}$$

$$\mathbf{m}_1 \cdot \mathbf{m}_2 = \mathbf{n}_1 \cdot \mathbf{n}_2. \tag{2.103}$$

An immediate consequence of this result is that orthogonal transformations in $\mathcal{R}^{n+1,1}$ induce angle-preserving transformations in $\mathcal{R}^n$. These are the conformal transformations discussed in the next section.

Relations among Three Points, Spheres or Hyperplanes

Let s_1, s_2, s_3 be three distinct nonzero vectors of $\mathcal{R}^{n+1,1}$ with non-negative square. Then the sign of

$$\Delta = s_1 \cdot s_2 \; s_2 \cdot s_3 \; s_3 \cdot s_1 \tag{2.104}$$

is invariant under the rescaling $s_1, s_2, s_3 \to \lambda_1 s_1, \lambda_2 s_2, \lambda_3 s_3$, where the λ's are nonzero scalars. Geometrically, when $s_i^2 > 0$, then $\widetilde{s}_i$ represents either a sphere or a hyperplane; when $s_i^2 = 0$, then s_i represents either a finite point or the point at infinity e. So the sign of Δ describes some geometric relationship among points, spheres or hyperplanes. Here we give a detailed analysis of the case when $\Delta < 0$.

When the s's are all null vectors, then $\Delta < 0$ is always true, as long as no two of them are linearly dependent.

When $s_1 = e$, s_2 is null, and $s_3^2 > 0$, then $\Delta < 0$ implies $\widetilde{s}_3$ to represent a sphere. Our previous analysis shows that $\Delta < 0$ if and only if the point s_2 is outside the sphere $\widetilde{s}_3$.

When s_1, s_2 are finite points and $s_3^2 > 0$, a simple analysis shows that $\Delta < 0$ if and only if the two points by s_1, s_2 are on the same side of the sphere or hyperplane $\widetilde{s}_3$.

When $s_1 = e$, $s_2^2, s_3^2 > 0$, then $\Delta < 0$ implies $\widetilde{s}_2$, $\widetilde{s}_3$ to represent two spheres. For two spheres with centers $\mathbf{c}_1, \mathbf{c}_2$ and radii ρ_1, ρ_2 respectively, we say they are (1) *near* if $|\mathbf{c}_1 - \mathbf{c}_2|^2 < \rho_1^2 + \rho_2^2$, (2) *far* if $|\mathbf{c}_1 - \mathbf{c}_2|^2 > \rho_1^2 + \rho_2^2$, and (3) *orthogonal* if $|\mathbf{c}_1 - \mathbf{c}_2|^2 = \rho_1^2 + \rho_2^2$. According to the first equation of (2.6), $\Delta < 0$ if and only if the two spheres $\widetilde{s}_2$ and $\widetilde{s}_3$ are far.

When s_1 is a finite point and $s_2^2, s_3^2 > 0$, then

- if $\widetilde{s}_2$ and $\widetilde{s}_3$ are hyperplanes, then $\Delta < 0$ implies that they are neither orthogonal nor identical. When the two hyperplanes are parallel, then $\Delta < 0$ if and only if the point s_1 is between the two hyperplanes. When the hyperplanes intersect, then $\Delta < 0$ if and only if s_1 is in the wedge domain of the acute angle in $\mathcal{R}^n$ formed by the two intersecting hyperplanes.

- if $\widetilde{s}_2$ is a hyperplane and $\widetilde{s}_3$ is a sphere, then $\Delta < 0$ implies that they are non-orthogonal, i.e., the center of the sphere does not lie on the hyperplane. If the center of a sphere is on one side of a hyperplane, we also say that the sphere is on that side of the hyperplane. If the point s_1 is outside the sphere $\widetilde{s}_3$, then $\Delta < 0$ if and only if s_1 and the sphere $\widetilde{s}_3$ are on the same side of the hyperplane $\widetilde{s}_2$; if the point is inside the sphere $\widetilde{s}_3$, then $\Delta < 0$ if and only if the point and the sphere are on opposite sides of the hyperplane.

- if $\widetilde{s}_2, \widetilde{s}_3$ are spheres, then $\Delta < 0$ implies that they are non-orthogonal. If they are far, then $\Delta < 0$ if and only if the point s_1 is either inside both of them or outside both of them. If they are near, then $\Delta < 0$ if and only if s_1 is inside one sphere and outside the other.

When s_1, s_2, s_3 are all of positive square, then $\Delta < 0$ implies that no two of them are orthogonal or identical.

- If they are all hyperplanes, with normals $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ respectively, then $\Delta < 0$ implies that no two of them are parallel, as the sign of Δ equals that of $\mathbf{n}_1 \cdot \mathbf{n}_2\ \mathbf{n}_2 \cdot \mathbf{n}_3\ \mathbf{n}_3 \cdot \mathbf{n}_1$. $\Delta < 0$ if and only if a normal vector of $\widetilde{s}_1$ with its base point at the intersection of the two hyperplanes $\widetilde{s}_2$ and $\widetilde{s}_3$, has its end point in the wedge domain of the acute angle in $\mathcal{R}^n$ formed by the two intersecting hyperplanes.

- If $\widetilde{s}_1$, $\widetilde{s}_2$ are hyperplanes and $\widetilde{s}_3$ is a sphere, then when the hyperplanes are parallel, $\Delta < 0$ if and only if the sphere's center is between the two hyperplanes. When the hyperplanes intersect, $\Delta < 0$ if and only if the sphere's center is in the wedge domain of the acute angle in $\mathcal{R}^n$ formed by the two intersecting hyperplanes.

- If $\widetilde{s}_1$ is a hyperplane and $\widetilde{s}_2, \widetilde{s}_3$ are spheres, then when the two spheres are far, $\Delta < 0$ if and only if the spheres are on the same side of the hyperplane. When the spheres are near, $\Delta < 0$ if and only if they are on opposite sides of the hyperplane.

- If all are spheres, then either they are all far from each other, or two spheres are far and the third is near to both of them.

Bunches of Spheres and Hyperplanes

In previous sections, we proved that Minkowski subspaces of $\mathcal{R}^{n+1,1}$ represent spheres and planes of various dimensions in $\mathcal{R}^n$. In this subsection we consider subspaces of $\mathcal{R}^{n+1,1}$ containing only their normals, which are vectors of positive square. Such subspaces are dual to Minkowski hyperspaces that represent spheres or hyperplanes. Therefore the tangent blade for a subspace A_r of $\mathcal{R}^{n+1,1}$ can be used to represent a set of spheres and hyperplanes, where each of them is represented by a vector of positive square. Or dually, the dual of A_r represents the intersection of a set of spheres and hyperplanes.

The simplest example is a pencil of spheres and hyperplanes. Let $\widetilde{s}_1, \widetilde{s}_2$ be two different spheres or hyperplanes, then the *pencil* of spheres/hyperplanes determined by them is the set of spheres/hyperplanes $(\lambda_1 s_1 + \lambda_2 s_2)^\sim$, where λ_1, λ_2 are scalars satisfying

$$(\lambda_1 s_1 + \lambda_2 s_2)^2 > 0. \tag{2.105}$$

The entire pencil is represented by the blade $A_2 = s_1 \wedge s_2$ or its dual $(s_1 \wedge s_2)^\sim$. There are three kinds of pencils corresponding to the three possible signatures of the blade $s_1 \wedge s_2$:

1. *Euclidean*, $(s_1 \wedge s_2)^2 < 0$. The space $(s_1 \wedge s_2)^\sim$, which is a subspace of any of the spaces $(\lambda_1 s_1 + \lambda_2 s_2)^\sim$, is Minkowski, and represents an $(n-2)$-dimensional sphere or plane in $\mathcal{R}^n$. If the point at infinity e is in the space, then the pencil $(s_1 \wedge s_2)^\sim$ is composed of hyperplanes passing through an $(n-2)$-dimensional plane. We call it a *concurrent pencil*.

   If e is not in the space $(s_1 \wedge s_2)^\sim$, there is an $(n-2)$-dimensional sphere that is contained in every sphere or hyperplane in the pencil $(s_1 \wedge s_2)^\sim$. We call it an *intersecting pencil*.

2. *Degenerate*, $(s_1 \wedge s_2)^2 = 0$. The space $(s_1 \wedge s_2)^\sim$ contains a one-dimensional null subspace, spanned by $P_{s_1}^\perp(s_2)$. If e is in the space, then the pencil is composed of hyperplanes parallel to each other. We call it a *parallel pencil*.

   If e is not in the space $(s_1 \wedge s_2)^\sim$, the pencil is composed of spheres tangent to each other at the point in $\mathcal{R}^n$ represented by the null vector $P_{s_1}^\perp(s_2)$. We call it a *tangent pencil*.

3. *Minkowski*, $(s_1 \wedge s_2)^2 > 0$. The Minkowski plane $s_1 \wedge s_2$ contains two non-collinear null vectors $|s_1 \wedge s_2| s_1 \pm |s_1||s_2| P_{s_1}^\perp(s_2)$. The two one-dimensional null spaces spanned by them are conjugate with respect to any of the vectors $\lambda_1 s_1 + \lambda_2 s_2$, which means that the two points represented by the two null vectors are inversive with respect to any sphere or hyperplane in the pencil $(s_1 \wedge s_2)^\sim$.

   If e is in the space $s_1 \wedge s_2$, then the pencil is composed of spheres centered at the point represented by the other null vector in the space. We call it a *concentric pencil*.

   If e is outside the space $s_1 \wedge s_2$, the two points represented by the two null vectors in the space are called *Poncelet points*. The pencil now is composed of spheres and hyperplanes with respect to which the two points are inversive. We call it a *Poncelet pencil*.

This finishes our classification of pencils. From the above analysis we also obtain the following corollary:

- The concurrent (or intersecting) pencil passing through an $(n-2)$-dimensional plane (or sphere) represented by Minkowski subspace A_n is $\widetilde{A}_n$.

- The parallel pencil containing a hyperplane $\widetilde{s}$ is $(e \wedge s)^{\sim}$. In particular, the parallel pencil normal to vector $\mathbf{n} \in \mathcal{R}^n$ is $(e \wedge \mathbf{n})^{\sim}$.

- The tangent pencil containing a sphere or hyperplane $\widetilde{s}$ and having tangent point $\mathbf{a} = P_E^{\perp}(a) \in \mathcal{R}^n$ is $(a \wedge s)^{\sim}$. In particular, the tangent pencil containing a hyperplane normal to $\mathbf{n} \in \mathcal{R}^n$ and having tangent point $\mathbf{a}$ is $(a \wedge (\mathbf{n} + \mathbf{a} \cdot \mathbf{n} e))^{\sim}$.

- The concentric pencil centered at $\mathbf{a} = P_E^{\perp}(a) \in \mathcal{R}^n$ is $(e \wedge a)^{\sim}$.

- The Poncelet pencil with Poncelet points $\mathbf{a}, \mathbf{b} \in \mathcal{R}^n$ is $(a \wedge b)^{\sim}$.

The generalization of a pencil is a bunch. A *bunch* of spheres and hyperplanes determined by r spheres and hyperplanes $\widetilde{s}_1, \ldots, s_r^{\sim}$ is the set of spheres and hyperplanes $(\lambda_1 s_1 + \cdots + \lambda_r s_r)^{\sim}$, where the λ's are scalars and satisfy

$$(\lambda_1 s_1 + \cdots + \lambda_r s_r)^2 > 0. \tag{2.106}$$

When $s_1 \wedge \cdots \wedge s_r \neq 0$, the integer $r-1$ is called the *dimension* of the bunch, and the bunch is represented by $(s_1 \wedge \cdots \wedge s_r)^{\sim}$. A pencil is a one-dimensional bunch. The dimension of a bunch ranges from 1 to $n-1$.

The classification of bunches is similar to that of pencils. Let $(s_1 \wedge \cdots \wedge s_r)^{\sim}$, $2 \leq r \leq n$, be a bunch. Then the signature of the space $(s_1 \wedge \cdots \wedge s_r)^{\sim}$ has three possibilities:

1. *Minkowski.* The space $(s_1 \wedge \cdots \wedge s_r)^{\sim}$ corresponds to an $(n-r)$-dimensional sphere or plane of $\mathcal{R}^n$, and is contained in any of the spheres and hyperplanes $(\lambda_1 s_1 + \cdots + \lambda_r s_r)^{\sim}$.

   If e is in the space, then the bunch is composed of hyperplanes passing through an $(n-r)$-dimensional plane. We call it a *concurrent bunch.* If e is not in the space, there is an $(n-r)$-dimensional sphere that are on any sphere or hyperplane in the bunch. We call it an *intersecting bunch.*

2. *Degenerate.* The space $(s_1 \wedge \cdots \wedge s_r)^{\sim}$ contains a one-dimensional null subspace, spanned by the vector $(s_1 \wedge \cdots \wedge s_r) \cdot (s_1 \wedge \cdots \wedge \check{s}_i \wedge \cdots \wedge s_r)$, where the omitted vector s_i is chosen so that $(s_1 \wedge \cdots \wedge \check{s}_i \wedge \cdots \wedge s_r)^2 \neq 0$.

   If e is in the space $(s_1 \wedge \cdots \wedge s_r)^{\sim}$, then the bunch is composed of hyperplanes normal to an $(r-1)$-space of $\mathcal{R}^n$ represented by the blade $e_0 \cdot (s_1 \wedge \cdots \wedge s_r)$. We call it a *parallel bunch.* If e is not in the space, the bunch is composed of spheres and hyperplanes passing through a point $\mathbf{a}_i \in \mathcal{R}^n$ represented by the null vector of the space, at the same time orthogonal to the $(r-1)$-plane of $\mathcal{R}^n$ represented by $e \wedge a \wedge (e \cdot (s_1 \wedge \cdots \wedge s_r))$. We call it a *tangent bunch.*

3. *Euclidean.* The Minkowski space $s_1 \wedge \cdots \wedge s_r$ corresponds to an $(r-2)$-dimensional sphere or plane. It is orthogonal to all of the spheres and hyperplanes $(\lambda_1 s_1 + \cdots + \lambda_r s_r)^{\sim}$.

   If e is in the space $s_1 \wedge \cdots \wedge s_r$, then the pencil is composed of hyperplanes perpendicular to the $(r-2)$-plane represented by $s_1 \wedge \cdots \wedge s_r$, together

Geometric conditions	Bunch A_r	Bunch $\widetilde{A}_r$
$A_r \cdot A_r^\dagger < 0\,,$ $e \wedge A_r = 0$	Concurrent bunch, concurring at the $(r-2)$-plane A_r	Concentric bunch, centered at the $(r-2)$-plane A_r
$A_r \cdot A_r^\dagger < 0\,,$ $e \wedge A_r \neq 0$	Intersecting bunch, at the $(r-2)$-sphere A_r	Poncelet bunch, with Poncelet sphere A_r
$A_r \cdot A_r^\dagger = 0\,,$ $e \wedge A_r = 0$	Parallel bunch, normal to the $(n-r+1)$-space $(e_0 \cdot A_r)^\sim$	Parallel bunch, normal to the $(r-1)$-space $e \wedge e_0 \wedge (e_0 \cdot A_r)$
$A_r \cdot A_r^\dagger = 0$, $e \wedge A_r \neq 0$, assuming a is a null vector in the space A_r	Tangent bunch, at point a and orthogonal to the $(n-r+1)$-plane $(e \cdot A_r)^\sim$	Tangent bunch, at point $\mathbf{a}$ and orthogonal to the $(r-1)$-plane $e \wedge a \wedge (e \cdot A_r)$
$A_r \cdot A_r^\dagger > 0\,,$ $e \wedge A_r = 0$	Concentric bunch, centering at the $(n-r)$-plane $\widetilde{A}_r$	Concurrent bunch, concurring at the $(n-r)$-plane $\widetilde{A}_r$
$A_r \cdot A_r^\dagger > 0\,,$ $e \wedge A_r \neq 0$	Poncelet bunch, with Poncelet sphere $\widetilde{A}_r$	Intersecting bunch, at the $(n-r)$-sphere $\widetilde{A}_r$

Table 2.1: Bunch dualities

with spheres whose centers are in the $(r-2)$-plane. We call it a *concentric bunch*. If e is outside the space, the $(r-2)$-sphere represented by $s_1 \wedge \cdots \wedge s_r$ is called a *Poncelet sphere*. Now the pencil is composed of spheres and hyperplanes orthogonal to the Poncelet sphere, called a *Poncelet bunch*.

Finally we discuss duality between two bunches. Let A_r, $2 \leq r \leq n$, be a blade. Then it represents an $(n-r+1)$-dimensional bunch. Its dual, $\widetilde{A}_r$, represents an $(r-1)$-dimensional bunch. Any bunch and its dual bunch are orthogonal, i.e., any sphere or hyperplane in a bunch A_r is orthogonal to a sphere or hyperplane in the bunch $\widetilde{A}_r$. Table 2.1 provides details of the duality.

2.7 Conformal Transformations

A transformation of geometric figures is said to be *conformal* if it *preserves shape*; more specifically, it preserves angles and hence the shape of straight lines and circles. As first proved by Liouville [L1850] for $\mathcal{R}^3$, any conformal transformation on the whole of $\mathcal{R}^n$ can be expressed as a composite of *inversions* in spheres and *reflections* in hyperplanes. Here we show how the homogeneous model of $\mathcal{E}^n$ simplifies the formulation of this fact and thereby facilitates computations with conformal transformations. Simplification stems from the fact that the conformal group on $\mathcal{R}^n$ is isomorphic to the Lorentz group on $\mathcal{R}^{n+1}$. Hence nonlinear conformal transformations on $\mathcal{R}^n$ can be linearized by representing them as Lorentz transformation and thereby further simplified as versor representations. The present treatment follows, with some improvements, [H91], where more details can be found.

From Chapter 1, we know that any Lorentz transformation $\underline{G}$ of a generic point $x \in \mathcal{R}^{n+1}$ can be expressed in the form

$$\underline{G}(x) = Gx(G^*)^{-1} = \sigma x' , \tag{2.107}$$

where G is a versor and σ is a scalar. We are only interested in the action of $\underline{G}$ on homogeneous points of $\mathcal{E}^n$. Since the null cone is invariant under $\underline{G}$, we have $(x')^2 = x^2 = 0$. However, for fixed e, $x \cdot e$ is not Lorentz invariant, so a scale factor σ has been introduced to ensure that $x' \cdot e = x \cdot e = -1$ and x' remains a point in $\mathcal{E}^n$. Expressing the right equality in (2.107) in terms of homogeneous points we have the expanded form

$$G[\mathbf{x} + \tfrac{1}{2}\mathbf{x}^2 e + e_0](G^*)^{-1} = \sigma[\mathbf{x}' + \tfrac{1}{2}(\mathbf{x}')^2 e + e_0] , \tag{2.108}$$

where

$$\mathbf{x}' = g(\mathbf{x}) \tag{2.109}$$

is a conformal transformation on $\mathcal{R}^n$ and

$$\sigma = -e \cdot (\underline{G}x) = -\langle\, eG^* x G^{-1} \,\rangle . \tag{2.110}$$

We study the simplest cases first.

For reflection by a vector $s = -s^*$ (2.107) becomes

$$\underline{s}(x) = -sxs^{-1} = x - 2(s \cdot x)s^{-1} = \sigma x' , \tag{2.111}$$

where $sx + xs = 2s \cdot x$ has been used. Both inversions and reflections have this form as we now see by detailed examination.

Inversions. We have seen that a circle of radius ρ centered at point $c = \mathbf{c} + \frac{1}{2}\mathbf{c}^2 e + e_0$ is represented by the vector

$$s = c - \tfrac{1}{2}\rho^2 e . \tag{2.112}$$

We first examine the important special case of the unit sphere centered at the origin in $\mathcal{R}^n$. Then s reduces to $e_0 - \frac{1}{2}e$, so $-2s \cdot x = \mathbf{x}^2 - 1$ and (2.111) gives

$$\sigma x' = (\mathbf{x} + \tfrac{1}{2}\mathbf{x}^2 e + e_0) + (\mathbf{x}^2 - 1)(e_0 - \tfrac{1}{2}e) = \mathbf{x}^2[\mathbf{x}^{-1} + \tfrac{1}{2}\mathbf{x}^{-2}e + e_0] . \tag{2.113}$$

Whence the inversion

$$g(\mathbf{x}) = \mathbf{x}^{-1} = \frac{1}{\mathbf{x}} = \frac{\mathbf{x}}{|\,\mathbf{x}\,|^2} . \tag{2.114}$$

Type	$g(\mathbf{x})$ on $\mathcal{R}^n$	Versor in $\mathcal{R}_{n+1,1}$	$\sigma(\mathbf{x})$
Reflection	$-\mathbf{nxn}+2\mathbf{n}\delta$	$s=\mathbf{n}+e\,\delta$	1
Inversion	$\dfrac{\rho^2}{\mathbf{x}-\mathbf{c}}+\mathbf{c}$	$s=c-\frac{1}{2}\rho^2 e$	$\left(\dfrac{\mathbf{x}-\mathbf{c}}{\rho}\right)^2$
Rotation	$R(\mathbf{x}-\mathbf{c})R^{-1}+\mathbf{c}$	$R_{\mathbf{c}}=R+e(\mathbf{c}\times R)$	1
Translation	$\mathbf{x}-\mathbf{a}$	$T_{\mathbf{a}}=1+\frac{1}{2}\mathbf{a}e$	1
Transversion	$\dfrac{\mathbf{x}-\mathbf{x}^2\mathbf{a}}{\sigma(\mathbf{x})}$	$K_{\mathbf{a}}=1+\mathbf{a}e_0$	$1-2\mathbf{a}\cdot\mathbf{x}+\mathbf{x}^2\mathbf{a}^2$
Dilation	$\lambda\mathbf{x}$	$D_\lambda=e^{-\frac{1}{2}E\ln\lambda}$	λ^{-1}
Involution	$\mathbf{x}^*=-\mathbf{x}$	$E=e\wedge e_0$	-1

Table 2.2: Conformal transformations and their versor representations (see text for explanation)

Note how the coefficient of e_0 has been factored out on the right side of (2.113) to get $\sigma=\mathbf{x}^2$. This is usually the best way to get the rescaling factor, rather than separate calculation from (2.110). Actually, we seldom care about σ, but it must be factored out to get the proper normalization of $g(\mathbf{x})$.

Turning now to inversion with respect to an arbitrary circle, from (2.112) we get

$$s\cdot x=c\cdot x-\tfrac{1}{2}\rho^2 e\cdot x=-\tfrac{1}{2}[(\mathbf{x}-\mathbf{c})^2-\rho^2]\,. \tag{2.115}$$

Insertion into (2.111) and a little algebra yields

$$\sigma x'=\left(\frac{\mathbf{x}-\mathbf{c}}{\rho}\right)^2\left[g(\mathbf{x})+\tfrac{1}{2}[g(\mathbf{x})]^2 e+e_0\right], \tag{2.116}$$

where

$$g(\mathbf{x})=\frac{\rho^2}{\mathbf{x}-\mathbf{c}}+\mathbf{c}=\frac{\rho^2}{(\mathbf{x}-\mathbf{c})^2}(\mathbf{x}-\mathbf{c})+\mathbf{c} \tag{2.117}$$

is the inversion in $\mathcal{R}^n$.

Reflections. We have seen that a hyperplane with unit normal $\mathbf{n}$ and signed distance δ from the origin in $\mathcal{R}^n$ is represented by the vector

$$s=\mathbf{n}+e\,\delta\,. \tag{2.118}$$

Inserting $s \cdot x = \mathbf{n} \cdot \mathbf{x} - \delta$ into (2.111) we easily find

$$g(\mathbf{x}) = \mathbf{n}\mathbf{x}\mathbf{n}^* + 2\mathbf{n}\delta = \mathbf{n}(\mathbf{x} - \mathbf{n}\delta)\mathbf{n}^* + \mathbf{n}\delta\,. \tag{2.119}$$

We recognize this as equivalent to a reflection $\mathbf{n}\mathbf{x}\mathbf{n}^*$ at the origin translated by δ along the direction of $\mathbf{n}$. A point $\mathbf{c}$ is on the hyperplane when $\delta = \mathbf{n} \cdot \mathbf{c}$, in which case (2.118) can be written

$$s = \mathbf{n} + e\,\mathbf{n} \cdot \mathbf{c}\,. \tag{2.120}$$

Via (2.119), this vector represents reflection in a hyperplane through point $\mathbf{c}$.

With a minor exception to be explained, all the basic conformal transformations in Table 2.2 can be generated from inversions and reflections. Let us see how.

Translations. We have already seen in Chapter 1 that versor $T_{\mathbf{a}}$ in Table 2.2 represents a translation. Now notice

$$(\mathbf{n} + e\,\delta)\mathbf{n} = 1 + \tfrac{1}{2}\mathbf{a}e = T_{\mathbf{a}} \tag{2.121}$$

where $\mathbf{a} = 2\delta\mathbf{n}$. This tells us that the composite of reflections in two parallel hyperplanes is a translation through twice the distance between them.

Transversions. The transversion $K_{\mathbf{a}}$ in Table 2.2 can be generated from two inversions and a translation; thus, using $e_0 e e_0 = -2e_0$ from (2.5c) and (2.5d), we find

$$e_+ T_{\mathbf{a}} e_+ = (\tfrac{1}{2}e - e_0)(1 + \tfrac{1}{2}\mathbf{a}e)(\tfrac{1}{2}e - e_0) = 1 + \mathbf{a}e_0 = K_{\mathbf{a}}\,. \tag{2.122}$$

The transversion generated by $K_{\mathbf{a}}$ can be put in the various forms

$$g(\mathbf{x}) = \frac{\mathbf{x} - \mathbf{x}^2\mathbf{a}}{1 - 2\mathbf{a} \cdot \mathbf{x} + \mathbf{x}^2\mathbf{a}^2} = \mathbf{x}(1 - \mathbf{a}\mathbf{x})^{-1} = (\mathbf{x}^{-1} - \mathbf{a})^{-1}\,. \tag{2.123}$$

The last form can be written down directly as an inversion followed by a translation and another inversion as asserted by (2.122). That avoids a fairly messy computation from (2.108).

Rotations. Using (2.120), the composition of reflections in two hyperplanes through a common point $\mathbf{c}$ is given by

$$(\mathbf{a} + e\,\mathbf{a} \cdot \mathbf{c})(\mathbf{b} + e\,\mathbf{b} \cdot \mathbf{c}) = \mathbf{a}\mathbf{b} + e\,\mathbf{c} \cdot (\mathbf{a} \wedge \mathbf{b})\,, \tag{2.124}$$

where $\mathbf{a}$ and $\mathbf{b}$ are unit normals. Writing $R = \mathbf{a}\mathbf{b}$ and noting that $\mathbf{c} \cdot (\mathbf{a} \wedge \mathbf{b}) = \mathbf{c} \times R$, we see that (2.124) is equivalent to the form for the rotation versor in Table 2.2 that we found in Chapter 1. Thus we have established that the product of two reflections at any point is equivalent to a rotation about that point.

Dilations. Now we prove that the composite of two inversions centered at the origin is a dilation (or dilatation). Using (2.5d) we have

$$(e_0 - \tfrac{1}{2}e)(e_0 - \tfrac{1}{2}\rho^2 e) = \tfrac{1}{2}(1 - E) + \tfrac{1}{2}(1 + E)\rho^2\,. \tag{2.125}$$

Normalizing to unity and comparing to (2.6) with $\rho = e^{\varphi}$, we get

$$D_\rho = \tfrac{1}{2}(1 + E)\rho + \tfrac{1}{2}(1 - E)\rho^{-1} = e^{E\varphi}\,, \tag{2.126}$$

where D_ρ is the square of the versor form for a dilation in Table 2.2. To verify that D_ρ does indeed generate a dilation, we note from (2.8) that

$$\underline{D}_\rho(e) = D_\rho e D_\rho^{-1} = D_\rho^2 e = \rho^{-2} e\,.$$

Similarly

$$\underline{D}_\rho(e_0) = \rho^2 e_0\,.$$

Therefore,

$$D_\rho(\mathbf{x} + \tfrac{1}{2}\mathbf{x}^2 e + e_0)D_\rho^{-1} = \rho^2[\rho^{-2}\mathbf{x} + \tfrac{1}{2}(\rho^{-2}\mathbf{x})^2 e + e_0]\,. \tag{2.127}$$

Thus $g(\mathbf{x}) = \rho^{-2}\mathbf{x}$ is a dilation as advertised.

We have seen that every vector with positive signature in $\mathcal{R}^{n+1,1}$ represents a sphere or hyperplane as well as an inversion or reflection in same. They compose a multiplicative group which we identify as the versor representation of the full *conformal group* $\mathrm{C}(n)$ of $\mathcal{E}^n$. Subject to a minor proviso explained below, our construction shows that this conformal group is equivalent to the *Lorentz group* of $\mathcal{R}^{n+1,1}$. Products of an even number of these vectors constitute a subgroup, the spin group $\mathrm{Spin}^+(n+1,1)$. It is known as the spin representation of the *proper Lorentz group*, the orthogonal group $\mathrm{O}^+(n+1,1)$. This, in turn, is equivalent to the *special orthogonal group* $\mathrm{SC}^+(n+1,1)$.

Our constructions above show that translations, transversions, rotations, dilations belong to $\mathrm{SC}^+(n)$. Moreover, every element of $\mathrm{SC}^+(n)$ can be generated from these types. This is easily proved by examining our construction of their spin representations $T_\mathbf{a}$, $K_\mathbf{b}$, $R_\mathbf{c}$, D_λ from products of vectors. One only needs to show that every other product of two vectors is equivalent to some product of these. Not hard! Comparing the structure of $T_\mathbf{a}$, $K_\mathbf{b}$, $R_\mathbf{c}$, D_λ exhibited in Table 2.2 with equations (2.6) through (2.15b), we see how it reflects the structure of the Minkowski plane $\mathcal{R}_{1,1}$ and groups derived therefrom.

Our construction of $\mathrm{Spin}^+(n+1,n)$ from products of vectors with positive signature excludes the bivector $E = e_+e_-$ because $e_-^2 = -1$. Extending $\mathrm{Spin}^+(n+1,n)$ by including E we get the full spin group $\mathrm{Spin}(n+1,n)$. Unlike the elements of $\mathrm{Spin}^+(n+1,n)$, E is not parametrically connected to the identity, so its inclusion gives us a double covering of $\mathrm{Spin}^+(n+1,n)$. Since E is a versor, we can ascertain its geometric significance from (2.108); thus, using (2.5c) and (2.5a), we easily find

$$E(\mathbf{x} + \tfrac{1}{2}\mathbf{x}^2 e + e_0)E = -[\,-\mathbf{x} + \tfrac{1}{2}\mathbf{x}^2 e + e_0\,]\,. \tag{2.128}$$

This tells us that E represents the *main involution* $\mathbf{x}^* = -\mathbf{x}$ of $\mathcal{R}_n$, as shown in Table 2.2. The conformal group can be extended to include involution, though this is not often done. However, in even dimensions involution can be achieved by a rotation so the extension is redundant.

Including E in the versor group brings all vectors of negative signature along with it. For $e_- = Ee_+$ gives us one such vector, and $D_\lambda e_-$ gives us (up to scale factor) all the rest in the E-plane. Therefore, extension of the versor group corresponds only to extension of $\mathrm{C}(n)$ to include involution.

Since every versor G in $\mathcal{R}_{n+1,1}$ can be generated as a product of vectors, expression of each vector in the expanded form (2.17) generates the expanded form

$$G = e(-e_0 A + B) - e_0(C + e\,D) \tag{2.129}$$

where A, B, C, D are versors in $\mathcal{R}_n$ and a minus sign is associated with e_0 for algebraic convenience in reference to (2.5d) and (2.2a). To enforce the versor property of G, the following conditions must be satisfied

$$AB^\dagger,\ BD^\dagger,\ CD^\dagger,\ AC^\dagger \in \mathcal{R}^n\,, \tag{2.130}$$

$$GG^\dagger = AD^\dagger - BC^\dagger = \pm|\,G\,|^2 \neq 0\,. \tag{2.131}$$

Since G must have a definite parity, we can see from (2.129) that A and D must have the same parity which must be opposite to the parity of C and D. This implies that the products in (2.130) must have odd parity. The stronger condition that these products must be vector-valued can be derived by generating G from versor products or from the fact that the conformal transformation generated by G must be vector-valued. For $G \in \mathrm{Spin}^+(n+1,n)$ the sign of (2.131) is always positive, but for $G \in \mathrm{Spin}(n+1,n)$ a negative sign may derive from a vector of negative signature.

Adopting the normalization $|\,G\,| = 1$, we find

$$G^{*\dagger} = \pm(G^*)^{-1} = -(A^{*\dagger}e_0 + B^{*\dagger})e + (C^{*\dagger} - D^{*\dagger}e)e_0\,, \tag{2.132}$$

and inserting the expanded form for G into (2.108), we obtain

$$g(\mathbf{x}) = (A\mathbf{x} + B)(C\mathbf{x} + D)^{-1} \tag{2.133}$$

with the rescaling factor

$$\sigma = \sigma_g(\mathbf{x}) = (C\mathbf{x} + D)(C^*\mathbf{x} + D^*)^\dagger\,. \tag{2.134}$$

In evaluating (2.131) and (2.110) to get (2.134) it is most helpful to use the property $\langle\, MN\,\rangle = \langle\, NM\,\rangle$ for the scalar part of any geometric product.

The general *homeographic form* (2.133) for a conformal transformation on $\mathcal{R}^n$ is called a *Möbius transformation* by Ahlfors [A85]. Because of its nonlinear form it is awkward for composing transformations. However, composition can be carried out multiplicatively with the versor form (2.129) with the result inserted into the homeographic form. As shown in [H91], the versor (2.133) has a 2×2 matrix representation

$$[\,G\,] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\,, \tag{2.135}$$

so composition can be carried out by matrix multiplication. Ahlfors [A86] has traced this matrix representation back to Vahlen [V02].

The apparatus developed in this section is sufficient for efficient formulation and solution of any problem or theorem in conformal geometry. As an example, consider the problem of deriving a conformal transformation on the whole of $\mathcal{R}^n$ from a given transformation of a finite set of points.

Let $\mathbf{a}_1, \cdots, \mathbf{a}_{n+2}$ be distinct points in $\mathcal{R}^n$ spanning the whole space. Let $\mathbf{b}_1, \cdots, \mathbf{b}_{n+2}$ be another set of such points. If there is a Möbius transformation g changing $\mathbf{a}_i$ into $\mathbf{b}_i$ for $1 \leq i \leq n+2$, then g must be induced by a Lorentz transformation $\underline{G}$ of $\mathcal{R}^{n+1,1}$, so the corresponding homogeneous points are related by

$$\underline{G}(a_i) = \lambda_i b_i, \qquad \text{for} \qquad 1 \leq i \leq n+2\,. \tag{2.136}$$

Therefore $a_i \cdot a_j = (\lambda_i b_i) \cdot (\lambda_j b_j)$ and g exists if and only if the λ's satisfy

$$(\mathbf{a}_i - \mathbf{a}_j)^2 = \lambda_i \lambda_j (\mathbf{b}_i - \mathbf{b}_j)^2\,.\ \text{for}\ 1 \leq i \neq j \leq n+2, \tag{2.137}$$

This sets $(n+2)(n-1)/2$ constraints on the $\mathbf{b}$'s from which the λ's can be computed if they are satisfied.

Now assuming that g exists, we can employ (2.136) to compute $g(\mathbf{x})$ for a generic point $\mathbf{x} \in \mathcal{R}^n$. Using the a_i as a basis, we write

$$x = \sum_{i=1}^{n+1} x^i \mathbf{a}_i, \tag{2.138}$$

so $\underline{G}(x) = \sum_{i=1}^{n+1} x^i \lambda_i b_i$, and

$$g(\mathbf{x}) = \frac{\sum_{i=1}^{n+1} x^i \lambda_i \mathbf{b}_i}{\sum_{i=1}^{n+1} x^i \lambda_i}. \tag{2.139}$$

The x's can be computed by employing the basis dual to $\{a_i\}$ as explained in Chapter 1.

If we are given, instead of $n+2$ pairs of corresponding points, two sets of points, spheres and hyperplanes, say $t_1, \cdots, t_{n+2}$, and $u_1, \cdots, u_{n+2}$, where $t_i^2 \geq 0$ for $1 \leq i \leq n+2$ and where both sets are linearly independent vectors in $\mathcal{R}^{n+1,1}$, then we can simply replace the $\mathbf{a}$'s with the t's and the $\mathbf{b}$'s with the u's to compute g.

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